Solutions to HW 9

1. (# 7.37) If so then given $q \in \mathbb{Q}$ $\exists$ open interval $(q-\varepsilon, q+\varepsilon) \cap \mathbb{Q}$ and compact $F \subset \mathbb{Q} \ni (q-\varepsilon, q+\varepsilon) \cap \mathbb{Q} \subset F$. Since $\mathbb{Q}$ is T_2 then F is closed. However, $(q-\varepsilon, q+\varepsilon) \cap \mathbb{I}$ are limit points that are not in $\mathbb{Q}$ $\Rightarrow$ F not closed. $\divideontimes$

2. Letting $X = \mathbb{Z}^+ \cup \{\infty\}$ and $Y = \{\frac{1}{n}\} \cup \{0\}$ consider the mapping $f: X \to Y$ defined as

$$f(x) = \begin{cases} 1/x & \text{if } x \in \mathbb{Z}^+ \\ 0 & \text{if } x = \infty \end{cases}$$

This is 1-1 and onto, so it remains to prove f & f^{-1} are cont. First note that we have the following bases:

T_X : The subspace top on $\mathbb{Z}^+$ is the discrete top and so the compact subsets of $\mathbb{Z}^+$ are (only) those that are finite. A basis for T_X therefore consists of singletons and sets of the form ~~$\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\} \cup$~~ $\{k, k+1, k+2, \ldots\} \cup \{\infty\}$.

T_Y : Using a similar argument a basis consists of singletons $\{\frac{1}{n}\}$ and sets of the form $\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\} \cup \{0\}$

Given that f maps basis sets to basis sets (and same with f^{-1}) it follows that f is a homeomorphism.

3. In the proof the fact that compact sets in a T_2 space are closed was needed (twice). Without T_2 then one needs to explicitly require that A is also closed.

4. a) Yes, because $x_1, x_2 \in Y \Rightarrow x_1, x_2 \in X \Rightarrow \exists$ open U_1, U_2 with $x_i \in U_i$ & $U_1 \cap U_2 = \emptyset \Rightarrow U_1 - x_0$ and $U_2 - x_0$ are open, disjoint neighborhoods of x_1 & x_2 (respectively)

b) $X = [0,1]$ & $x_0 = 0 \Rightarrow Y = (0,1]$

As an example of an open cover that does not have a finite subcover, let $\mathcal{O}_Y = \{ (\frac{1}{n}, 1) : n \in \mathbb{Z}^+ \} \cup \{ (\frac{1}{2}, \frac{3}{2}) \cap X \}$

X must be infinite because finite sets are always compact

c) Given $y \in Y$ then $y \neq x_0 \Rightarrow \exists$ open $U_x, U_y \ni x_0 \in U_x$, $y \in U_y$ & $U_x \cap U_y = \emptyset \Rightarrow U_y \subset Y$ is an open neighborhood of y. Also, $\overline{U_y}$ is a closed subset of $X \Rightarrow \overline{U_y}$ closed and $x_0 \notin \overline{U_y} \Rightarrow y \in U_y \subset \overline{U_y} \subset Y$.

5. ~~[illegible]~~ you must also include X connected

6. Assume $f : X \to Y$ is a homeom. and X is locally compact. Given $y \in Y$ then $\exists!\, x \in X \ni f(x) = y \Rightarrow \exists$ open U_x and compact $F_x \ni x \in U_x \subset F_x \subset X \Rightarrow y \in f(U_x) \subset f(F_x) \subset Y$, where $f(U_x)$ is open (since f^{-1} is cont) and $f(F_x)$ is compact (since compact is a top. property)