Solutions to HW8

7.1 Since $A = \phi$ is compact, assume that $A \neq \phi$. Given an open cover $\mathcal{O}_A$ of A pick $A_0 \in \mathcal{O}_A$. Since A_0 is open then $\mathbb{R} - A_0$ is finite $\Rightarrow$ finite number of open sets from $\mathcal{O}_A$ are needed to cover these points $\Rightarrow$ $\exists$ a finite subcover $\Rightarrow$ A compact

7.3 The sets $A_n = [\frac{1}{n}, 1]$ are compact but $\cup A_n = (0,1]$ is not

7.8 If not then $\{C_i^c\}$ is an open cover for X [pf: C_i^c is open $\forall i$ with $\cup C_i^c \subset X$, and $(\cup C_i^c)^c = \cap C_i = \phi \Rightarrow \cup C_i^c = X$]. Since X is compact $\exists$ finite subcover $C_{i_1}^c, C_{i_2}^c, \ldots, C_{i_k}^c$ with $i_j < i_{j+1}$ (so $C_{i_{j+1}} \subset C_{i_j}$). With this $C_{i_k}^c = X \Rightarrow C_{i_k} = \phi$ $\divideontimes$

7.11 a) It suffices to prove that f^{-1} is continuous. We do this using closed sets (rather than open sets). So $A \subset X$ closed $\Rightarrow$ (since X is compact) A is compact $\Rightarrow f(A)$ compact $\Rightarrow f(A)$ closed $\Rightarrow f^{-1}$ cont.

b) Let $f(x) = x$, $X = \mathbb{R}$ (with st. top) and $Y = \mathbb{R}$ (with trivial top)

c) Let $f(x) = x$, $X = \mathbb{R}$ (with discrete top) and $Y = \mathbb{R}$ (with st. top)

7.15 The sets $[0, 1 - \frac{1}{n})$, for $n \in \mathbb{Z}^+$, and $[1,2)$ are open and $\mathcal{O} = \{[0, 1-\frac{1}{n}) : n \in \mathbb{Z}^+\} \cup [1,2)$ is an open cover of $[0,1]$ that has no finite subcover

7.24 If $A = \phi$ then $f_A = 0$, which is continuous. So, assume that $A \neq \phi$. Given $a \in A$, and $x, x_0 \in X$ we have the following inequalities

$\mathrm{dist}(x, A) \leq d(x, a) \leq d(x, x_0) + d(x_0, a)$
$\mathrm{dist}(x_0, A) \leq d(x_0, a) \leq d(x_0, x) + d(x, a)$

Since these hold $\forall\, a \in A$ we conclude that

$$d(x, A) \leq d(x, x_0) + d(x_0, A)$$
$$d(x_0, A) \leq d(x, x_0) + d(x, A)$$

$\Rightarrow$

$$| d(x, A) - d(x_0, A) | \leq d(x, x_0)$$

$\Rightarrow$

$$| f_A(x) - f_A(x_0) | \leq d(x, x_0)$$

So, given $\varepsilon > 0$ let $\delta = \varepsilon$ so that $d(x, x_0) < \delta$
$\Rightarrow | f_A(x) - f_A(x_0) | \leq \varepsilon \Rightarrow f_A$ cont $\forall\, x_0 \in X$

7.35 Letting $x_n = 1 - \frac{1}{n}$ then $\{x_n\}$ is not convergent in $\mathbb{R}_e$ nor does it contain a convergent subsequence