Solutions to HW 6

4.2 First note that $f^{-1}(A^c) = f^{-1}(A)^c$ [[pf: $f^{-1}(A^c) = \{x \in X : f(x) \notin A\} = \{x \in X : f(x) \in A\}^c = f^{-1}(A)^c$]].

So, f cont $\iff f^{-1}(C^c)$ open $\forall$ closed C
$\iff f^{-1}(C^c)^c$ closed $\forall$ closed C
$\iff f^{-1}(C)$ closed $\forall$ closed C

4.23 Letting $T_1 = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$
$T_2 = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}\}$
$T_3 = \{\emptyset, X, \{a,b\}, \{a,c\}, \{a\}\}$

T_1 & T_2 homeomorphic: $f: (X, T_1) \to (X, T_2)$
$f(a)=a,\ f(b)=c,\ f(c)=b$
$f^{-1}(a)=a,\ f^{-1}(c)=b,\ f^{-1}(b)=c$
$\therefore f$ & f^{-1} map open to open (& are 1-1 and onto)

T_3 not homeom. to either T_1 or T_2: not possible because $f: (X, T_1) \to (X, T_3)$ must map open to open and be 1-1

4.24 This follows if f & f^{-1} are closed maps, and this follows directly from Exercise 4.2

4.26 a) Letting $f(x) = b + (a-b)e^{-x}$ then $f: [0,\infty) \to [a,b]$ is 1-1 and onto [[reason: $f' = (b-a)e^{-x} > 0 \Rightarrow$ strictly increasing with $f(0)=a$ and $f(\infty)=b$]] and is continuous. The inverse fn $x = \ln\frac{b-a}{b-y} : [a,b] \to [0,\infty)$ has same properties $\therefore$ homeom.

b) Letting $f(x) = a + (b-a)e^{x} : (-\infty, 0] \to [a,b]$ is 1-1 and onto [[reason: $f' > 0 \Rightarrow$ strictly increasing with $f(-\infty) = a$ and $f(0) = b$]]. The inverse $x = \ln\frac{y-a}{b-a} : [a,b] \to (-\infty, 0]$ has same properties $\therefore$ homeo.

4.32 a) First note that because $f: X \to Y$ is a homeom. that every open set in X is uniquely identified with an open set in Y (and visa-versa). With this we have the following

$$\mathring{A} = \bigcup_{\substack{U \subset A \\ U \text{ open}}} U \Rightarrow f(\mathring{A}) = f(\cup U)$$

$$= \cup f(U) = \mathring{f(A)}$$

b) Because the closed sets ~~in~~ X & Y are in correspondence we get that

$$\bar{A} = \bigcap_{\substack{A \subset B \\ B \text{ closed}}} B \Rightarrow f(\bar{A}) = f(\cap B) \quad f \text{ 1-1}$$

$$= \cap f(B) = \overline{f(A)}$$

c) $f(\partial A) = f(\bar{A} - \mathring{A}) \underset{f \text{ homeom.}}{=} f(\bar{A}) - f(\mathring{A}) = \overline{f(A)} - \mathring{f(A)} = \partial f(A)$

5.2 a) the checklist:

i) $d_M \geq 0$: because $|p_i - q_i| \geq 0 \ \forall i$

$d_M = 0 \Rightarrow |p_1 - q_1| = |p_2 - q_2| = 0 \Rightarrow p = q$

$p = q \Rightarrow p - q = 0 \Rightarrow |p_i - q_i| = 0 \Rightarrow d_M = 0$

ii) $d_M(p,q) = d_M(q,p)$: follows because $|p_i - q_i| = |q_i - p_i|$

iii) this follows because

$$|p_i - q_i| = |p_i - z_i + z_i - q_i| \leq |p_i - z_i| + |z_i - q_i|$$

b) if $p = (1,1)$ & $q = (1,2)$ then $d(p,q) = 0$ but $p \neq q$

5.5 The checklist:

i) clearly $d_M \geq 0$ and $d_M = 0 \iff x = y$

ii) given that $x \neq y \iff y \neq x$ then this follows

iii) this follows by examining the various values for $d(x,y)$, $d(x,z)$ & $d(z,y)$ [eg. $d(x,y) = 0$ then clearly $d(x,y) \leq d(x,z) + d(z,y)$, and if $d(x,y) = 1$ (so $x \neq y$) then either $x \neq z$ or $z \neq y$ and from this it follows that $d(x,y) = 1 \leq d(x,z) + d(z,y)$]

5.9 This follows directly from Th 1.9