Solutions to HW4

2.13 e) $[-1,1]$

f) $[-1,1)$

2.18 $\{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\}$

2.23 a) $\phi \overset{?}{\in} T$: yes, by definition

$\mathbb{R} \overset{?}{\in} T$: yes, because $\mathbb{R}^c = \phi$ is countable

$U_1, U_2 \in T \overset{?}{\Rightarrow} U_1 \cap U_2 \in T$:
$(U_1 \cap U_2)^c = U_1^c \cup U_2^c$ } finite union of countable is countable

$U_\alpha \in T \overset{?}{\Rightarrow} \cup U_\alpha \in T$: $(\cup U_\alpha)^c = \cap U_\alpha^c \subset U_\alpha \Rightarrow$ countable

b) if $0 \in U \in T$ then
$U \cap A = U \cap (\mathbb{R} - \{0\}) = U \cap (\mathbb{R} \cap \{0\}^c)$
$= (U \cap (-\infty, 0)) \cup (U \cap (0, \infty))$
if $U \cap A = \phi$ then $U \cap (-\infty, 0) = \phi$ & $U \cap (0, \infty) = \phi$
$\Rightarrow U = \{0\} \Rightarrow U^c = \mathbb{R} - \{0\}$ not countable
$\Rightarrow U$ not open $\Rightarrow U \cap A \neq \phi$
$\Rightarrow \{0\}$ is a limit pt

c) Assume $x_n \in \mathbb{R}$ with $x_n \to 0 \Rightarrow \forall U \in T$ with $0 \in U$
$\exists N \ni x_n \in U \quad \forall n \geq N$
Letting $U = \mathbb{R} - \{x_n\}_{n=1}^{\infty}$, then $U \in T$ and $0 \in U$
and $\nexists N$ as required
$\therefore \nexists \{x_n\}$ with $x_n \to 0$

2.24 e) $\partial A = \{-1, 1, 2\}$

f) $\partial A = \{2\}$

2.28 a) ∂A closed

pf: $\partial A = \bar{A} - \mathring{A} = \bar{A} \cap (\mathring{A})^c$ } intersection of closed sets $\Rightarrow$ closed

b) $\partial A = \bar{A} \cap \overline{(X - A)}$

pf: since $\overline{X - A} = X - \mathring{A} = X \cap (\mathring{A})^c$

then

$\bar{A} \cap (X - \mathring{A})^c = \bar{A} \cap (\mathring{A})^c = \bar{A} - \mathring{A} = \partial A$

c) $\partial A \cap \mathring{A} = \emptyset$

pf: $\partial A \cap \mathring{A} = (\bar{A} \cap \mathring{A}^c) \cap \mathring{A} = \bar{A} \cap (\mathring{A}^c \cap \mathring{A})$

$= \bar{A} \cap \emptyset = \emptyset$

d) $\partial A \cup \mathring{A} = \bar{A}$

pf: $\partial A \cup \mathring{A} = (\bar{A} \cap \mathring{A}^c) \cup \mathring{A}$

$= (\bar{A} \cup \mathring{A}) \cap (\mathring{A}^c \cup \mathring{A})$

$= \bar{A} \cap X = \bar{A}$

e) $\partial A \subset A \iff A$ closed

pf: ($\Leftarrow$) A closed $\Rightarrow \partial A = \bar{A} - \mathring{A} = A - \mathring{A} \subset A$

($\Rightarrow$) $\partial A \subset A \Rightarrow \bar{A} = \partial A \cup \mathring{A} \subset A \cup \mathring{A} = A \Rightarrow A$ closed

f) $\partial A \cap A = \emptyset \iff A$ open

pf: ($\Leftarrow$) A open $\Rightarrow \partial A \cap A = \partial A \cap \mathring{A} = \emptyset$

($\Rightarrow$) $\partial A \cap A = \emptyset$

A not open $\Rightarrow \exists\, x \in A$ $\forall$ open ~~and open~~ U_i with $x \in U_i$

$U_i \cap A^c \neq \emptyset$. Given that $U_i \cap A \neq \emptyset$

then $x \in \partial A \Rightarrow \partial A \cap A \neq \emptyset$ $*$

g) $\partial A = \emptyset \iff A$ open & closed

pf: ($\Rightarrow$) $\partial A = \emptyset$

1) $\partial A \subset A \Rightarrow A$ closed

2) $\partial A \cap A = \emptyset \Rightarrow A$ open

($\Leftarrow$) $\partial A = \bar{A} - \mathring{A} = \emptyset - \emptyset = \emptyset$