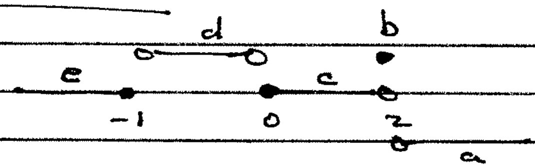
Solutions to HW10

1. (#3.23)



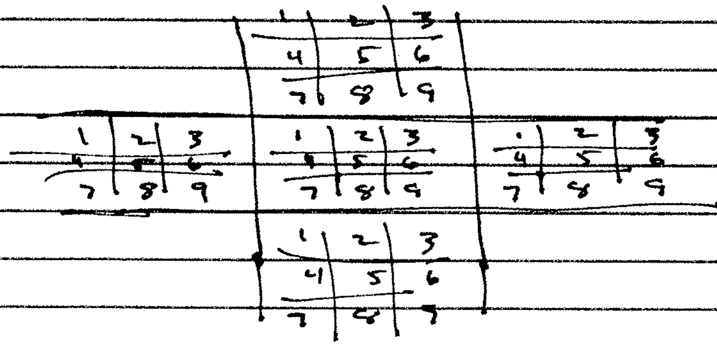
$Y = \{a, b, c, d, e\}$.

It is assumed that a, b, c, d, & e are not equal.

a) A basis is $\mathcal{T}_* = \{\emptyset, Y, \{a\}, \{d\}, \{a,d\}, \{c,d\}, \{e,d\}, \{e,d,c\}, \{a,b,c,d\}\}$

b) In addition to the sets in part (a) add $\{c\}$, $\{a,b\}$, $\{c,b,d\}$

2. (# 3.26)



The torus makes the tic-tac-toe periodic (as shown in diagram)

Any 3 (together) in columns or rows win:
123, or 312, or 231 etc
Any 3 (together) along a diagonal win:
726, 483, etc (the two stated are new)

3. a) $U \in \mathcal{T}_Y \Rightarrow f^{-1}(U) \in \mathcal{T}_X \Rightarrow U \in \mathcal{T}_f \Rightarrow \mathcal{T}_Y \subset \mathcal{T}_f$

As an example of when $\mathcal{T}_Y \neq \mathcal{T}_f$ let $f: X \to Y$ where $X = \mathbb{R}$, $\mathcal{T}_X =$ st. top, $\mathcal{T}_Y =$ trivial top and

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

So, $Y = \{0, 1\}$ and $\mathcal{T}_f = \{\emptyset, Y, 0\}$ while $\mathcal{T}_Y = \{\emptyset, Y\}$

b) It suffices to prove that $\mathcal{T}_f \subset \mathcal{T}_Y$. So

$U \in \mathcal{T}_f \underset{\text{quot. map}}{\Rightarrow} f^{-1}(U) \in \mathcal{T}_X \underset{\text{open map}}{\Rightarrow} f(f^{-1}(U)) \in \mathcal{T}_Y \Rightarrow U \in \mathcal{T}_Y$

4. Letting $f: X \to Y$ and $g: Y \to Z$, where $f(X) = Y$, then $U \in T_Z \Rightarrow g^{-1}(U) \in T_Y \Rightarrow f^{-1}(g^{-1}(U)) \in T_X$

5. (# 14.1)

a) if the point is in $\mathbb{R}$ the result follows immediately, so assume that $x_0 = p_e$. Letting

$$f(x) = \begin{cases} x & \text{if } x \neq p_e \\ 0 & \text{if } x = p_e \end{cases}$$

This is 1-1 and onto so it remains to prove f & f^{-1} are continuous.

f cont: $f^{-1}(\alpha,\beta) = \begin{cases} (\alpha,\beta) & \text{if } \beta<0 \text{ or } \alpha>0 \\ (\alpha,0)\cup p_e \cup (0,\beta) & \text{otherwise}\end{cases}$

both sets are open $\Rightarrow$ f cont

f^{-1} cont: $f(a,b) = (a,b)$ and
$f((c,0) \cup p_e \cup (0,d)) = (c,d)$
both sets are open so f^{-1} is cont

b) Take internal endpoints (which produces a countable basis)

6. (# 14.2)

T_2: $\mathbb{R}_d$ & $\mathbb{R}$ T_2 $\Rightarrow$ $\mathbb{R}_d \times \mathbb{R}$ is T_2

$\mathbb{R}'$ neighborhood: given $(x,y) \in \mathbb{R}_d \times \mathbb{R}$ let $f(x,y) = y$. In this case $f: x \times (y-\varepsilon, y+\varepsilon) \to (y-\varepsilon, y+\varepsilon)$ is a homeom.

non-countable basis: any basis of $\mathbb{R}_d \times \mathbb{R}$ must include sets of form $\{x\} \times U$ where $x \in \mathbb{R}$ and U is an open set in $\mathbb{R}$ (U from a basis for st. top). ~~Since~~ If one assumes the basis is countable it must include all $x \in \mathbb{R}$ (which is an uncountable set), and this is not possible.

7. (# 14.4) If one is orientable (say S_1) and one not (S_2) then $\exists R_2 \subset S_2$ and homeom. $g: R_2 \to M$ (where $M =$ Möbius band). If $R_1 = f^{-1}(R_2)$ then $h = f \circ g : R_1 \to M$ a homeo $\Rightarrow$ S_1 not orientable ⚹

let $f: S_1 \to S_2$ be a homeom.