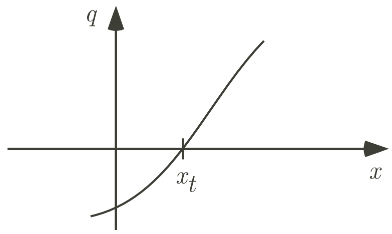

Figure 4.3 In the analysis of the turning point, $q(x)$ is assumed to have a simple zero at $x = x_t$ with $q'(x_t) > 0$. This will enable us to use a linear approximation for $q(x)$ near the turning point (see (4.30)).

In this problem, $0 < x < 1$ and $g(x, \eta) = \mu_0 x + \lambda(\eta)$, where $\mu_0 > 0$ is constant and $\lambda(\eta) > 0$ is continuous with $\lambda \to \infty$ as $\eta \to \infty$. Find a first-term approximation of the solution of this problem for small ε.

4.3 Turning Points

We will introduce the analysis for turning points using the example from the previous section. The equation to solve is therefore

$$\varepsilon^2 y'' - q(x)y = 0. \tag{4.25}$$

As pointed out in deriving the WKB approximation, we must stay away from the points where $q(x)$ is zero. To explain how to deal with these points, we will assume that $q(x)$ is smooth and has a simple zero at $x = x_t$. In other words, we will assume that $q(x_t) = 0$ and $q'(x_t) \neq 0$. To start we will consider what happens when there is only one such point.

4.3.1 The Case of When $q'(x_t) > 0$

We are assuming here that there is a simple turning point at x_t, with $q(x) > 0$ if $x > x_t$ and $q(x) < 0$ if $x < x_t$ (see Figure 4.3). This means that the solution of (4.25) will be oscillatory if $x < x_t$ and exponential for $x > x_t$. The fact that the solution is oscillatory for negative $q(x)$ can be understood if one considers the constant coefficient equation $y'' + \lambda^2 y = 0$, where $\lambda > 0$. The general solution in this case is $y(x) = A_0 \cos(\lambda x + \phi_0)$ and this clearly is oscillatory. A similar explanation can be given for the exponential solutions for a positive coefficient.

We can use the WKB approximation on either side of the turning point. This gives the following approximation for the general solution

$$y \sim \begin{cases} y_L(x, x_t) & \text{if} \quad x < x_t, \\ y_R(x, x_t) & \text{if} \quad x_t < x, \end{cases} \tag{4.26}$$

where

$$y_R(x, x_t) = \frac{1}{q(x)^{1/4}} \left[a_R \exp\left(-\frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} ds \right) + b_R \exp\left(\frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} ds \right) \right], \tag{4.27}$$

and

$$y_L(x, x_t) = \frac{1}{q(x)^{1/4}} \left[a_L \exp\left(-\frac{1}{\varepsilon} \int_{x}^{x_t} \sqrt{q(s)} ds \right) + b_L \exp\left(\frac{1}{\varepsilon} \int_{x}^{x_t} \sqrt{q(s)} ds \right) \right]. \tag{4.28}$$

These expressions come directly from (4.11) except that we have now fixed one of the endpoints in the integrals at the turning point. This particular choice is not mandatory, but it does simplify the formulas in the calculations to follow. It is important to recognize that the coefficients in (4.27) and (4.28) are not independent, and we must find out how they are connected by investigating what is happening in a transition layer centered at $x = x_t$. This is very similar to what happened when we were studying interior-layer problems in Chapter 2. When we finish, the approximation of the general solution will only contain two arbitrary constants rather than four as in (4.27) and (4.28).

Solution in Transition Layer

To determine the solution near the turning point we introduce the transition layer coordinate

$$\bar{x} = \frac{x - x_t}{\varepsilon^{\beta}}, \tag{4.29}$$

or equivalently

$$x = x_t + \varepsilon^{\beta} \bar{x}.$$

We know the point x_t but we will have to determine the value for β from the analysis to follow. Now, to transform the differential equation, we first use Taylor's theorem to conclude

$$\begin{aligned} q(x_t + \varepsilon^{\beta} \bar{x}) &\sim q(x_t) + \varepsilon^{\beta} \bar{x} q'(x_t) + \cdots \\ &= \varepsilon^{\beta} \bar{x} q'(x_t) + \cdots . \end{aligned} \tag{4.30}$$

We will assume $q(x)$ has a simple zero at x_t, so $q'(x_t) \neq 0$. With this, and letting $Y(\bar{x})$ denote the solution in this layer, we have that

$$\varepsilon^{2-2\beta} Y'' - (\varepsilon^{\beta} \bar{x} q_t' + \cdots) Y = 0, \tag{4.31}$$

where $q_t' = q'(x_t)$. For balancing, we need $2 - 2\beta = \beta$, and so $\beta = \frac{2}{3}$. The appropriate expansion of the solution in this region is

$$Y \sim \varepsilon^{\gamma} Y_0(\bar{x}) + \cdots . \tag{4.32}$$

Introducing this into (4.31), we get the following equation to solve

$$Y_0'' - \bar{x} q_t' Y_0 = 0, \quad \text{for } -\infty < \bar{x} < \infty. \tag{4.33}$$

This can be transformed into an equation whose solutions are known. Letting $s = (q_t')^{1/3}\bar{x}$, we get *Airy's equation* which is

$$\frac{d^2}{ds^2} Y_0 - s Y_0 = 0, \quad \text{for } -\infty < s < \infty. \tag{4.34}$$

Because of its importance in applied mathematics, Airy's equation has been studied extensively. It can be solved using power series expansions or the Laplace transform. One finds that the general solution can be written as

$$Y_0 = a\,\text{Ai}(s) + b\,\text{Bi}(s),$$

where $\text{Ai}(\cdot)$ and $\text{Bi}(\cdot)$ are Airy functions of the first and second kinds, respectively, and a and b are arbitrary constants. The definitions and some of the properties of these functions are given in Appendix B. We are now able to write the general solution of the transition layer equation (4.33) as

$$Y_0(\bar{x}) = a\,\text{Ai}[(q_t')^{1/3}\bar{x}] + b\,\text{Bi}[(q_t')^{1/3}\bar{x}]. \tag{4.35}$$

From (4.27), (4.28) and (4.35) we have six undetermined constants. However, the solution in (4.35) must match with the outer solutions in (4.27) and (4.28). This will lead to connection formulas between the constants, and these will result in two arbitrary constants in the general solution.

Matching

The solution in the transition region must match with the outer solutions given in (4.27) and (4.28). In what follows, keep in mind that $q_t' > 0$. To do the matching, we will use the intermediate variable

$$x_\eta = \frac{x - x_t}{\varepsilon^\eta}, \tag{4.36}$$

where $0 < \eta < 2/3$. Before matching the solutions, note that the terms in the two outer solutions contain the following: for $x > x_t$

$$\begin{aligned}\int_{x_t}^{x} \sqrt{q(s)}ds &= \int_{x_t}^{x_t+\varepsilon^\eta x_\eta} \sqrt{q(s)}ds \\ &\sim \int_{x_t}^{x_t+\varepsilon^\eta x_\eta} \sqrt{(s-x_t)q_t'}ds \\ &= \frac{2}{3}\varepsilon r^{3/2},\end{aligned} \tag{4.37}$$

and

$$\begin{aligned}q(x)^{-1/4} &\sim [q_t + (x-x_t){q_t}']^{-1/4} \\ &= \varepsilon^{-1/6}(q_t')^{-1/6} r^{-1/4},\end{aligned} \tag{4.38}$$

where $r = ({q_t}')^{1/3}\varepsilon^{\eta-2/3}x_\eta$.

Matching for $x > x_t$

Using the asymptotic expansions for the Airy functions given in Appendix B, one finds that

$$\begin{aligned}Y &\sim \varepsilon^\gamma Y_0(\varepsilon^{\eta-2/3}x_\eta) + \cdots \\ &\sim \frac{a\varepsilon^\gamma}{2\sqrt{\pi}r^{1/4}}e^{-\frac{2}{3}r^{3/2}} + \frac{b\varepsilon^\gamma}{\sqrt{\pi}r^{1/4}}e^{\frac{2}{3}r^{3/2}},\end{aligned} \tag{4.39}$$

and for the WKB solution in (4.27)

$$y_R \sim \frac{\varepsilon^{-1/6}}{(q_t')^{1/6}r^{1/4}}\left(a_R e^{-\frac{2}{3}r^{3/2}} + b_R e^{\frac{2}{3}r^{3/2}}\right). \tag{4.40}$$

For these expressions to match, we must have $\gamma = -\frac{1}{6}$,

$$a_R = \frac{a}{2\sqrt{\pi}}(q_t')^{1/6} \quad \text{and} \quad b_R = \frac{b}{\sqrt{\pi}}(q_t')^{1/6}. \tag{4.41}$$

Matching for $x < x_t$

The difference with this case is that $x_\eta < 0$, which introduces complex numbers into (4.28). As before, using the expansions for the Airy functions given in the Appendix B,

$$\begin{aligned}Y &\sim \varepsilon^{\gamma} Y_0(\varepsilon^{\eta-2/3}x_\eta)\\ &\sim \frac{a\varepsilon^{\gamma}}{\sqrt{\pi}|r|^{1/4}}\cos\left(\frac{2}{3}|r|^{3/2}-\frac{\pi}{4}\right)+\frac{b\varepsilon^{\gamma}}{\sqrt{\pi}|r|^{1/4}}\cos\left(\frac{2}{3}|r|^{3/2}+\frac{\pi}{4}\right)\\ &= \frac{\varepsilon^{\gamma}}{2\sqrt{\pi}|r|^{1/4}}\left[(ae^{-i\pi/4}+be^{i\pi/4})e^{i\zeta}+(ae^{i\pi/4}+be^{-i\pi/4})e^{-i\zeta}\right], \qquad (4.42)\end{aligned}$$

where $\zeta = \frac{2}{3}|r|^{3/2}$. In the last step leading to (4.42), the identity $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$ was used. As for the WKB expansion, from (4.28),

$$y_L \sim \frac{\varepsilon^{-1/6}}{(q_t')^{1/6}|r|^{1/4}}\left(a_L e^{i(\zeta-\frac{\pi}{4})}+b_L e^{-i(\zeta+\frac{\pi}{4})}\right). \qquad (4.43)$$

Matching (4.42) and (4.44) yields the following

$$a_L = \frac{(q_t')^{1/6}}{2\sqrt{\pi}}(ia+b), \qquad (4.44)$$

and

$$b_L = \frac{(q_t')^{1/6}}{2\sqrt{\pi}}(a+ib) = i\bar{a}_L, \qquad (4.45)$$

where $\bar{a}_L$ is the complex conjugate of a_L. Equations (4.41), (4.45), and (4.46) are known as *connection formulas* and they constitute a system of equations that enable us to solve for four of the constants in terms of the remaining two.

Summary

Solving the connection formulas in (4.41), (4.45), and (4.46) it is found that

$$\mathbf{a}_L = \mathbf{M}\mathbf{a}_R, \qquad (4.46)$$

where

$$\mathbf{a}_L = \begin{pmatrix} a_L \\ b_L \end{pmatrix}, \qquad \mathbf{a}_R = \begin{pmatrix} a_R \\ b_R \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} i & \frac{1}{2} \\ 1 & \frac{1}{2}i \end{pmatrix}. \qquad (4.47)$$

The resulting WKB approximation in(4.26) can be written as

$$y(x) \sim \begin{cases} \dfrac{1}{|q(x)|^{1/4}}\left[2a_R\cos\left(\dfrac{1}{\varepsilon}\theta(x)-\dfrac{\pi}{4}\right)+b_R\cos\left(\dfrac{1}{\varepsilon}\theta(x)+\dfrac{\pi}{4}\right)\right] & \text{if } x<x_t\\ \dfrac{1}{q(x)^{1/4}}\left(a_R e^{-\kappa(x)/\varepsilon}+b_R e^{\kappa(x)/\varepsilon}\right) & \text{if } x_t<x, \end{cases} \qquad (4.48)$$

where

$$\theta(x) = \int_x^{x_t} \sqrt{|q(s)|}ds, \tag{4.49}$$

and

$$\kappa(x) = \int_{x_t}^{x} \sqrt{|q(s)|}ds. \tag{4.50}$$

It should be remembered that this expansion was derived under the assumption that $x = x_t$ is a simple turning point with $q'(x_t) > 0$. The accuracy of this approximation near the turning point depends on the specific problem. However, one can show that in general one must require $\varepsilon^{2/3} << |x - x_t|$ (see Exercise 4.11). Also, as expected, we have ended up with an expansion for the solution of (4.25) that contains two arbitrary constants $(a_R,\, b_R)$.

Example

As an example of a turning point problem consider the following

$$\varepsilon^2 y'' = x(2-x)y, \quad \text{for } -1 < x < 1, \tag{4.51}$$

where $y(-1) = y(1) = 1$. In this case $q(x) = x(2-x)$, and so there is a simple turning point at $x = 0$ with $q'(0) = 2$. The WKB approximation in (4.49) therefore applies where, from (4.50) and (4.51),

$$\theta(x) = \frac{1}{2}(1-x)\sqrt{x(x-2)} - \frac{1}{2}\ln[1 - x + \sqrt{x(x-2)}],$$

and

$$\kappa(x) = \frac{1}{2}(x-1)\sqrt{x(2-x)} - \frac{1}{2}\arcsin(x-1) + \frac{\pi}{4}\,.$$

The constants a_R and b_R are found from the two boundary conditions. For example, since $q(1) = 1$ and $\kappa(1) = \pi/4$, then the condition $y(1) = 1$ leads to the equation $a_R \exp(-\frac{\pi}{4}\varepsilon) + b_R \exp(\frac{\pi}{4}\varepsilon) = 1$. The resulting approximation obtained from (4.49) is shown in Figure 4.4(a) along with the approximation from the transition layer. The singularity in y_R and y_L at $x = 0$ is evident in this plot as is the matching by the Airy functions from the transition layer. To demonstrate the accuracy of the WKB approximation even when a turning point is present, in Figure 4.4(b) the numerical solution is plotted along with the left and right WKB approximations. Within their respective domains of applicability, the WBK approximations are essentially indistinguishable from the numerical solution. ■

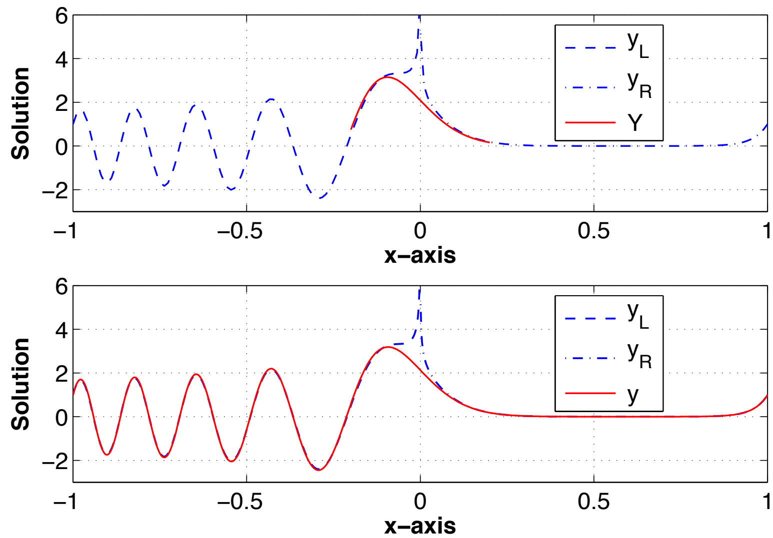

Figure 4.4 (a) The transition layer solution (4.32), and the left and right WKB approximations given in (4.49), for the solution of (4.52). (b) A comparison between the left and right WKB approximations and the numerical solution of (4.52). In these calculations $\varepsilon = \frac{1}{25}$.

4.3.2 The Case of When $q'(x_t) < 0$

This approximation derived for $q'(x_t) > 0$ can be used when $q'(x_t) < 0$ by simply making the change of variables $z = x_t - x$ (see Exercise 4.17). The result is that

$$y \sim \begin{cases} Y_L(x, x_t) & \text{if} \quad x < x_t, \\ Y_R(x, x_t) & \text{if} \quad x_t < x, \end{cases} \tag{4.52}$$

where

$$Y_R(x, x_t) = \frac{1}{q(x)^{1/4}} \left[A_R \exp\left(\frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} ds \right) + B_R \exp\left(-\frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} ds \right) \right], \tag{4.53}$$

and

$$Y_L(x, x_t) = \frac{1}{q(x)^{1/4}} \left[A_L \exp\left(\frac{1}{\varepsilon} \int_{x}^{x_t} \sqrt{q(s)} ds \right) + B_L \exp\left(-\frac{1}{\varepsilon} \int_{x}^{x_t} \sqrt{q(s)} ds \right) \right]. \tag{4.54}$$

The coefficients in these expressions satisfy the connection formula

$$\mathbf{A}_R = \mathbf{N}\mathbf{A}_L, \tag{4.55}$$

where

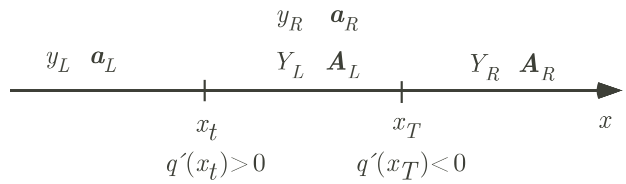

Figure 4.5 Schematic of the case of two turning points and the corresponding WKB approximations used in the various intervals. Also shown are the respective coefficient vectors.

$$\mathbf{A}_L = \begin{pmatrix} A_L \\ B_L \end{pmatrix}, \qquad \mathbf{A}_R = \begin{pmatrix} A_R \\ B_R \end{pmatrix}, \qquad \mathbf{N} = \begin{pmatrix} 1 & \frac{1}{2}i \\ i & \frac{1}{2} \end{pmatrix}. \tag{4.56}$$

Making use of the fact that $q'(x_t) < 0$, one can rewrite the approximation in this case as

$$y(x) \sim \begin{cases} \dfrac{1}{q(x)^{1/4}}\left(A_L e^{\theta(x)/\varepsilon} + B_L e^{-\theta(x)/\varepsilon}\right) & \text{if } x < x_t \\ \dfrac{1}{|q(x)|^{1/4}}\left[2A_L \cos\left(\dfrac{1}{\varepsilon}\kappa(x) - \dfrac{\pi}{4}\right) + B_L \cos\left(\dfrac{1}{\varepsilon}\kappa(x) + \dfrac{\pi}{4}\right)\right] & \text{if } x_t < x, \end{cases} \tag{4.57}$$

where $\theta(x)$ and $\kappa(x)$ are given in (4.50) and (4.51). Again, we have a solution that contains two arbitrary constants (A_L, B_L).

4.3.3 Multiple Turning Points

It is not particularly difficult to combine the above approximations to analyze problems with multiple turning points. To illustrate, suppose x_t and x_T, with $x_t < x_T$, are turing points with $q'(x_t) > 0$ and $q'(x_T) < 0$ (see Figure 4.5). Moreover, these are the only turing points. In this case, (4.26) holds for $x < x_T$. Similarly, (4.53) applies, with x_T replacing x_t, and this holds for $x > x_t$. What this means is that it is required that $y_R(x, x_t) = Y_L(x, x_T)$ for $x_t < x < x_T$. This holds if

$$\mathbf{a}_R = \mathbf{Q}\mathbf{A}_L, \tag{4.58}$$

where

$$\mathbf{Q} = \begin{pmatrix} e^{\phi} & 0 \\ 0 & e^{-\phi} \end{pmatrix}, \tag{4.59}$$

and

$$\phi = \frac{1}{\varepsilon}\int_{x_t}^{x_T} \sqrt{q(s)}ds. \tag{4.60}$$

Therefore,

$$y \sim \begin{cases} y_L(x,x_t) & \text{if} \quad x < x_t, \\ Y_L(x,x_T) & \text{if} \quad x_t < x < x_T, \\ Y_R(x,x_T) & \text{if} \quad x_T < x, \end{cases} \tag{4.61}$$

where $\mathbf{A}_R = \mathbf{N}\mathbf{A}_L$ and $\mathbf{a}_L = \mathbf{M}\mathbf{Q}\mathbf{A}_L$.

If, instead $q'(x_t) < 0$ and $q'(x_T) > 0$, then a similar analysis (see Exercise 4.17) shows that

$$y \sim \begin{cases} Y_L(x,x_t) & \text{if} \quad x < x_t, \\ y_L(x,x_T) & \text{if} \quad x_t < x < x_T, \\ y_R(x,x_T) & \text{if} \quad x_T < x, \end{cases} \tag{4.62}$$

where $\mathbf{a}_R = \mathbf{M}^{-1}\mathbf{a}_L$, $\mathbf{A}_L = \mathbf{N}^{-1}\mathbf{Q}^{-1}\mathbf{a}_L$, and $\mathbf{A}_R = \mathbf{Q}^{-1}\mathbf{a}_L$.

4.3.4 Uniform Approximation

In the case of a single turing point, with $q'(x_t) > 0$, the solution is in three pieces: the two WKB approximations (y_L, y_R) on either side of the turning point and the transition layer solution that is in-between. The first-term approximation of the solution in the transition layer was found to be a solution of Airy's equation (4.33), which is the prototype equation for a simple turning point. In fact, it is possible to find a uniform first-term expansion of the solution in terms of Airy functions. This was first derived by Langer [1931], and the result is

$$y(x) \sim \varepsilon^{-1/6}\left(\frac{f(x)}{q(x)}\right)^{1/4}\left[a_0\mathrm{Ai}\Big(\varepsilon^{-2/3}f(x)\Big) + b_0\mathrm{Bi}\Big(\varepsilon^{-2/3}f(x)\Big)\right], \tag{4.63}$$

where

$$f(x) = \begin{cases} \left[\dfrac{3}{2}\displaystyle\int_{x_t}^{x}\sqrt{q(s)}ds\right]^{2/3} & \text{if } x_t \le x \\ -\left[\dfrac{3}{2}\displaystyle\int_{x}^{x_t}\sqrt{-q(s)}ds\right]^{2/3} & \text{if } x \le x_t. \end{cases}$$

The derivation of (4.64) is outlined in Exercise 4.25. Also, in connection with the WKB approximations given in (4.49),

$$a_R = \frac{a_0}{2\sqrt{\pi}} \quad \text{and} \quad b_R = \frac{b_0}{2\sqrt{\pi}}\,.$$

Because of the value of having a composite expansion, there have been numerous generalizations of Langer's result. A discussion of some of these can be found in Nayfeh [1973].

In referring to the turning point as simple, it is meant that $q(x)$ has a simple zero at $x = x_t$, i.e., $q(x_t) = 0$ but $q'(x_t) \neq 0$. Higher-order turning points do arise, and an example of one of second-order, at $x = 0$, is

$$\varepsilon^2 y'' - x^2 e^x y = 0. \tag{4.64}$$

The reason this is second order is simply that $q(x_t) = q'(x_t) = 0$ but $q''(x_t) \neq 0$. It is also possible to have one of fractional order (e.g., when $q(x) = x^{1/3} e^x$). The prototype equations in these cases are discussed in Appendix B. Other types do occur, such as logarithmic, although they generally are harder to analyze.

Another complication that can arise is that the position of the turning point may depend on ε. This can lead to coalescing turning points and an example of this is found in the equation

$$\varepsilon^2 y'' - (x - \varepsilon)(x + \varepsilon) y = 0. \tag{4.65}$$

This has simple turning points at $x_t = \pm\varepsilon$, but to the first-term approximation they look like one of second order at $x = 0$. Situations such as this are discussed in Steele [1976] and Dunster [1996].

Examples

1. Suppose the problem is

$$\varepsilon^2 y'' + \sin(x) y = 0, \quad \text{for } 0 < x < 2\pi,$$

where $y(0) = a$ and $y(2\pi) = b$. This has three turning points: $x_t = 0, \pi, 2\pi$. Because two of these are endpoints there will be two outer WKB approximations, one for $0 < x < \pi$ and one for $\pi < x < 2\pi$. Since $\frac{d}{dx}\sin(x) \neq 0$ at $x = \pi$ then the turning point $x_t = \pi$ is simple and is treated in much the same way as the one analyzed above. The solution in the transition layer at $x_t = 0$ will be required to satisfy the boundary condition $y(0) = a$, and the solution in the layer at $x_t = 2\pi$ will satisfy $y(2\pi) = b$. ■

2. The equation

$$\varepsilon^2 y'' + p(x) y' + q(x) y = 0 \tag{4.66}$$

differs from the one considered above because of the first derivative term. The WKB approximation of the general solution is given in Exercise 4.3. Unlike before, the turning points now occur when $p(x) = 0$. It is not a coincidence that these are also the points that can give rise to an interior layer, and an example of this is given in Section 2.5.1. ■

Exercises

4.16. Consider the boundary value problem

$$\varepsilon^2 y'' + x(x+3)^2 y = 0, \quad \text{for } a < x < b,$$

where $y(a) = \alpha$ and $y(b) = \beta$. Find a first-term WKB expansion of the solution in the case of when:
(a) $a = 0$, $\alpha = 0$, $b = 1$, $\beta = 1$.
(b) $a = -1$, $\alpha = 1$, $b = 0$, $\beta = 0$.
(c) $a = -1$, $\alpha = 0$, $b = 1$, $\beta = 1$.

4.17. This exercise concerns the derivation of the WKB approximation when one or more turing points are present.
(a) By making the change of variable $z = x_t - x$ in (4.25) derive (4.58) from (4.49).
(b) Derive (4.63).

4.18. In quantum mechanics one has to solve the time independent Schrödinger equation given as

$$\varepsilon^2 \psi'' - [V(x) - E]\psi = 0, \quad \text{for } -\infty < x < \infty, \tag{4.67}$$

where the wave function $\psi(x)$ is required to be bounded. Also, the potential $V(x)$ is a given, smooth function while the energy E is a non-negative constant that must be solved for along with the wave function. Assume here that the situation is as depicted in Figure 4.6, with $V_m < E$ where $V_m = \min_{-\infty<x<\infty} V(x)$. Also, $V(a) = V(b) = E$.
(a) Using (4.49) and (4.58) find a first-term approximation of the solution. From this conclude that for there to be a nonzero solution it must be that E satisfies

$$\int_a^b \sqrt{E - V(x)}dx = \varepsilon\pi\left(n + \frac{1}{2}\right), \quad \text{for } n = 0, 1, 2, 3, \ldots. \tag{4.68}$$

This is known as the WKB quantization condition.

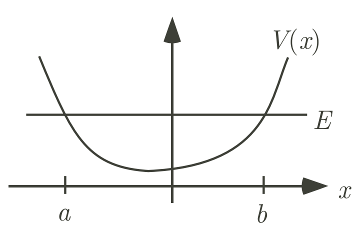

Figure 4.6 Potential used in Exercise 4.18.

(b) For a harmonic oscillator the potential is $V(x) = \mu x^2$ where $\mu > 0$ is constant. Solve the quantization condition (4.69) in this case. Also describe, in terms of nodes and wavelengths, what the integer n signifies for the solution in the region $a < x < b$. What does n signify for the solution outside this region? It is worth pointing out that for this potential the WKB quantization condition gives the exact values for E.

(c) In deriving the WKB solution in part (a), it was assumed that a, b, and E are independent of ε. However, the quantization condition (4.69) means they must depend on ε for there to be a nontrivial solution. How does this dependence affect the derivation of the WKB approximation? Also, what affect, if any, does the dependence of E on the integer n (which may be quite large) do to the accuracy of the approximation?

(d) In the case of when $V(x) = \mu x^m$, where $m = 2, 4, 6, \ldots$, one can show that $E^\gamma = \mu \varepsilon^m \chi$, where $\gamma = 1 + 2/m$ and χ satisfies the equation

$$\chi^{1/2} \sum_{k=0}^{\infty} a_k \chi^{-k} = 2n + 1, \tag{4.69}$$

where a_k is independent of χ [Dunham, 1932, Bender et al., 1977]. For example,

$$a_0 = \frac{2}{\sqrt{\pi}} \frac{\Gamma(1 + 1/m)}{\Gamma(3/2 + 1/m)} \quad \text{and} \quad a_1 = -\frac{m-1}{6\sqrt{\pi}} \frac{\cot(\pi/m)\Gamma(1/2 + 1/m)}{\Gamma(1/m)}.$$

Setting $N = 2n + 1$ and assuming $m \neq 2$, use (4.70) to find a two term expansion of E for $N >> 1$. How does the first-term compare with your result from part (a)?

(e) The numerical values of $E/\varepsilon^{m/\gamma}$ calculated by Secrest et al. [1962] are given in Table 4.2 in the case of when $V(x) = x^m$. Compare the results from parts (a) and (d) with these values. Make sure to comment on the accuracy as a function of n.

4.19. This problem concerns the Schrödinger equation in (4.68) and the potential shown in Figure 4.7. Because $V(x)$ acts like a barrier, this example is commonly used to illustrate the phenomenon of tunneling.

(a) Using (4.49) and (4.58) find a first-term approximation of the solution of (4.68) in the case of the barrier potential shown in Figure 4.7. Assume E

m/n	0	2	4
4	1.0603621	7.4556979	16.261826
8	1.2258201	10.244947	25.809007

Table 4.2 For Exercise 4.18(e)

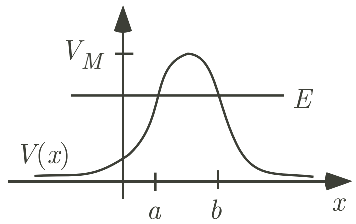

Figure 4.7 Potential used in Exercise 4.19.

is given and satisfies $0 < E < V_M$, where $V_M = \max_{-\infty<x<\infty} V(x)$ and $V(-\infty) = V(\infty) = 0$.

(b) The time-dependent Schrödinger equation has the form

$$-\varepsilon^2 \partial_x{}^2 \Psi + V\Psi = i\partial_t \Psi.$$

In regard to the barrier potential, suppose a wave approaches from the left. Because of reflection from the barrier, the solution should then consist of left and right traveling waves in the region $x < a$. However, part of the incident wave will be transmitted through the barrier and result in a right running wave for $x > b$ (this is the phenomenon of tunneling). Assuming $\Psi = \exp(-iEt)\psi(x)$, use the results from part (a) to find first-term approximations of the waves in these two regions.

4.20. The motion of planetary rings is described using the theory of self-gravitating annuli orbiting a central mass. For circular motion in the plane, with the planet at the origin, one ends up having to find the circumferential velocity $V(r, \theta, t) = v(r)e^{i(\omega t + m\theta)}$. The function $v(r)$ satisfies [Papaloizou and Pringle, 1987]

$$\frac{d}{dr}\left(r\frac{d}{dr}(rv)\right) = m^2(1 - \kappa^2 r^2)v, \quad \text{for } 0 < r < \infty,$$

where

$$\kappa = \frac{\alpha + \beta m}{m}.$$

Here r is the radial coordinate and α, β are positive constants. The parameter m is positive and is a mode number. Find a first-term approximation of the solution for large m.

4.21. Find a first-term approximation of the solution of

$$\varepsilon^2 y'' + xy' - x(1 + x)y = 0, \quad \text{for } -1 < x < 1,$$

where $y(-1) = 1$ and $y(1) = 3e^2$. [Hint: for large r,

$$\int_0^r e^{-\alpha s^2} ds \sim \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} - \frac{1}{2\alpha r} e^{-\alpha r^2}]$$

4.22. Find a first-term approximation of the solution of $\varepsilon^2 y'' - q(x)y = 0$ for $-\infty < x < \infty$, where $q(x)$ is the even function shown in Figure 4.8. Thus, $q(0) = q'(0) = 0$ but $q''(0) \neq 0$, and $q(\pm a) = 0$ with $q'(\pm a) \neq 0$.

4.23. Consider the problem

$$\varepsilon y'' + x^2 y' + y = 0, \quad \text{for } -1 < x < 1,$$

where $y(-1) = 1$ and $y(1) = -1$.

(a) The WKB/transition layer analysis based on the coordinate transformation (4.29) does not work on this problem. Explain why. Also, explain why failure should be expected based on the balancing in the transition layer and its relationship to the terms responsible for the singularity in the WKB expansion.

(b) Letting $r = (x + \varepsilon^\beta x_0)/\varepsilon^\alpha$, find the transition regions that are needed to complete the approximations. Also state the equations to be solved in each region and the matching conditions to be imposed.

(c) Show that it is possible to resolve the difficulty by making the substitution

$$y(x) = u(x) e^{-x^2/(6\varepsilon)}.$$

4.24. For Schrödinger's equation it has been observed that in certain cases it is possible to improve on the WKB quantization condition given in (4.69). In the SWKB method, one introduces what is known as a supersymmetric potential $W(x) \equiv -\varepsilon {\psi_0}'/\psi_0$, where $\psi_0(x)$ is an eigenfunction corresponding to the lowest energy E_0. Under certain restrictions, it is then possible to transform (4.69) into the following equation

$$\int_a^b \sqrt{E^* - W^2}\, dx = \varepsilon \pi n, \quad \text{for } n = 0, 1, 2, 3, \ldots, \tag{4.70}$$

where $E^* = E - E_0$. It is significant, and somewhat surprising, that for many of the cases of when the problem can be solved in closed form the condition in (4.71) actually yields the exact value for E [Comtet et al., 1985, Dutt et al., 1991]. In comparison, (4.69) has been found to be exact for very few cases.

(a) Show that $W(x)$ satisfies the equation $\varepsilon W' - W^2 = -V^*$, where $V^*(x) = V(x) - E_0$. From this and the WKB quantization condition (4.69), derive (4.71). To do this, remember that a and b are turning points and assume that $W(a) = -W(b)$.

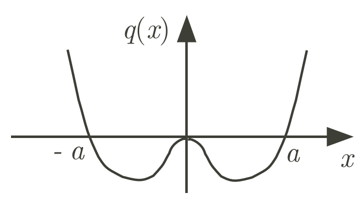

Figure 4.8 Potential used in Exercise 4.22.

(b) As another approach, introduce the supersymmetric potential into (4.68) and then derive a first-term approximation of the solution. With this, obtain the SWBK quantization condition (4.71).

4.25. This problem concerns the derivation of the uniformly valid approximation in (4.64).

(a) Change variables by letting $s = \varepsilon^{-\alpha} f(x)$ and $y(x) = \Phi(x)Y(s)$. Show that by taking $\Phi = f_x^{-1/2}$, where f satisfies $f f_x^2 = q$, that (4.1) can be transformed into an equation of the form $Y'' = [s + G(\varepsilon, s)]Y$.

(b) From the result in part (a), derive (4.64).

(c) Show (4.64) reduces to the approximation in (4.49) when $x >> x_t$ or when $x_t << x$. Does it reduce to (4.35) when $x \approx x_t$?

4.26. In the semiclassical description for what are called shape resonances [Combes et al., 1984], one finds the eigenvalue problem

$$\varepsilon^2 \psi'' - [V(x) - E]\psi = 0, \quad \text{for } 0 < x < \infty,$$

where $\psi(0) = \psi(\infty) = 0$. The potential here is $V(x) = (x-1)^2 \exp(-x^2/4)$. Find a first-term approximation of the solution, and from this derive the quantization condition.

4.4 Wave Propagation and Energy Methods

The WKB method is quite useful for finding an asymptotic approximation of a traveling wave solution of a linear partial differential equation. To illustrate this, we consider the problem

$$u_{xx} = \mu^2(x) u_{tt} + \alpha(x) u_t + \beta(x) u, \quad \text{for } \begin{cases} 0 < x < \infty \\ 0 < t, \end{cases} \tag{4.71}$$

where

$$u(0, t) = \cos(\omega t). \tag{4.72}$$

This is the equation for the displacement of a string that is damped (αu_t) and which has an elastic support (βu). Because the string is being forced periodically at the left end, the solution will develop into a wave that travels to the right. To find this solution, we will set $u(x, t) = e^{i\omega t} v(x)$ and then require that the function $v(x)$ is consistent with the observation that the wave moves to the right. Also, for the record, it is assumed that the functions in (4.72) are smooth with $\mu(x) > 0$, and $\alpha(x)$ and $\beta(x)$ non-negative.

The equation is linear, but there is no obvious small parameter ε that can serve as the basis of the WKB approximation. To motivate what will be done suppose $\alpha = \beta = 0$ and μ is constant. In this case, the plane wave solutions of (4.72), (4.73) are