Perturbation Methods
HW 8
Due May 11

1. The van der Pol equation is

$$\varepsilon y'' - (1 - y^2)y' + y = 0, \quad \text{for } \ t > 0.$$

Assume that $y(0) = \sqrt{3}$ and $y'(0) = 0$. Note that this is a modified version of Exercise 6.11.

(a) Letting $y = -v'$, show that the equation can be written as the first-order system

$$\begin{aligned} v' &= -y, \\ \varepsilon y' &= v + y - \frac{1}{3}y^3. \end{aligned}$$

What are the initial conditions? The numerical solution of this system is shown in Figure 1 for the case of when $\varepsilon = 10^{-3}$. The solution is periodic, and the solid curve in the graph is the trajectory through one cycle. Note that this path consists of two fast components, which are the horizontal segments, and two slow components, which follow the dashed (cubic) curve. Together they form what is known as a relaxation oscillation.

(b) Use the first term in the outer expansion of y and v to determine the equation of the cubic curve shown in Figure 1. Also use it to determine where the first corner layer is located (give the value of t, y, and v).

(c) Find the first term in the corner-layer expansion of y and v. With this, determine the location of the first transition layer.

(d) In each transition layer, the solution very quickly switches from one branch of the cubic to another. Because of this, the time the solution spends on the slow component of the cycle serves as a first-term approximation of the period. Use this to find an approximation of the period of the oscillation. For comparison, the period found numerically is $T = 1.91$ when $\varepsilon = 10^{-2}$ and $T = 1.68$ when $\varepsilon = 10^{-3}$).

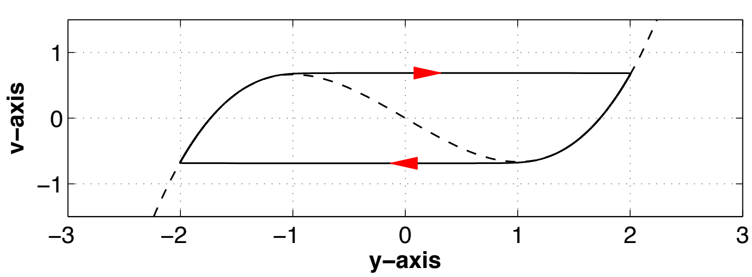

Figure 1: Diagram for Exercise 1.

2. 6.28

3. The Sel'kov model for the enzymatic driven reactions that result in the conversion of ATP into ADP is

$$y' = 1 - yv^{\gamma},$$
$$v' = \lambda v(yv^{\gamma-1} - 1),$$

where $\gamma > 1$ and $\lambda > 0$ are constants. In these equations, y and v designate the concentrations of ATP and ADP, respectively. Note that this is a modified version of Exercise 6.37.

(a) Find the steady-state and determine for what values of λ it is stable.

(b) Describe what happens near the point where the steady-state changes stability.

4. 6.47(a),(b)