Solutions to HW5

1. a) $t_1 = t,\ t_2 = \varepsilon^{\alpha} t$ & $y \sim \varepsilon^{\beta} y_0 + \varepsilon^{\gamma} y_1 + \cdots \qquad 0 < \beta < \gamma$

$$(1-e^{-y})e^{-y} = e^{-y} - e^{-2y} \qquad \& \quad y \ll 1$$
$$= 1 - y + \tfrac{1}{2}y^2 - \tfrac{1}{6}y^3 + \cdots - \left(1 - 2y + \tfrac{1}{2}\cdot 4y^2 - \tfrac{1}{6}\cdot 8y^3 + \cdots\right)$$
$$= y - \tfrac{3}{2}y^2 + \tfrac{7}{3}y^3 + \cdots$$

$\Rightarrow$

$$(\partial_1^2 + 2\varepsilon^{\alpha}\partial_1\partial_2 + \varepsilon^2\partial_2^2)(\varepsilon^{\beta}y_0 + \varepsilon^{\gamma}y_1 + \cdots)$$
$$+ \bar{\alpha}\cdot\varepsilon^2(\partial_1 + \varepsilon^{\alpha}\partial_2)(\varepsilon^{\beta}y_0 + \varepsilon^{\gamma}y_1 + \cdots)$$
$$+ \varepsilon^{\beta}y_0 + \varepsilon^{\gamma}y_1 + \cdots - \tfrac{3}{2}\left(\varepsilon^{2\beta}y_0^2 + 2\varepsilon^{\beta+\gamma}y_0y_1 + \cdots\right) + \tfrac{7}{3}\cdot\varepsilon^{3\beta}y_0^3 + \cdots$$
$$= \varepsilon^3\cos(1+\varepsilon^2\omega)t$$

$O(\varepsilon^{\beta})$ $\partial_1^2 y_0 + y_0 = 0 \quad \Rightarrow \quad y_0 = A(t_2)\cos[t_1 + \theta(t_2)]$

$A(0) = 0$

Balancing $\Rightarrow$ $\beta = 1,\ \alpha = 2$ & $\gamma = 2$

$\Omega = t_1 + \theta$

$O(\varepsilon^2)$ $\partial_1^2 y_1 + y_1 = \tfrac{3}{2}y_0^2 = \tfrac{3}{4}A^2\cdot[1 + \cos 2\Omega]$

$\Rightarrow$

$$y_1 = B(t_2)\cos[t_1 + \varphi(t_2)] + \tfrac{3}{4}A^2 - \tfrac{1}{4}A^2\cos 2\Omega$$

$O(\varepsilon^3)$ $\partial_1^2 y_2 + y_2 \underbrace{+ 2\partial_1\partial_2 y_0}_{1} \underbrace{+ \bar{\alpha}\,\partial_1 y_0}_{2} \underbrace{+ \tfrac{7}{6}y_0^3}_{3} \underbrace{- 3y_0y_1}_{4} = \underbrace{\cos(t_1 + \omega t_2)}_{5}$

$1 = 2\partial_2(-A\sin) = -2A'\sin - 2A\theta'\cos$

$2 = -\bar{\alpha}A\sin$

$3 = \tfrac{7}{6}A^3\cdot\tfrac{1}{4}[3\cos + \cos 3\Omega]$

$4 = -3A\cos\left(\tfrac{3}{4}A^2 - \tfrac{1}{4}A^2\cos 2\Omega\right) + \text{stuff}$

$= -\tfrac{9}{4}A^3\cos + \tfrac{3}{4}A^3\cdot\tfrac{1}{2}\cos + \text{stuff} = -\tfrac{15}{8}A^3\cos + \text{stuff}$

$5 = \cos(t_1 + \theta - \theta + \omega t_2) = \cos\cdot\cos(\theta - \omega t_2) + \sin\cdot\sin(\theta - \omega t_2)$

$\sin$: $-2A' - \bar{\alpha}A = \sin(\theta - \omega t_2)$

$\cos$: $-2A\theta' + \tfrac{7}{8}A^3 - \tfrac{15}{8}A^3 = \cos(\theta - \omega t_2)$

$\Rightarrow$

$$2A' + \bar{\alpha}A = -\sin(\theta - \omega t_2) \qquad A(0) = 0$$
$$2A\theta' + A^3 = -\cos(\theta - \omega t_2) \qquad \theta(0) = -\tfrac{\pi}{2}$$

$A \to \text{const} \Rightarrow \quad \theta - \omega t_2 = \text{const} \Rightarrow \theta = \omega t_2 + \theta_\infty$

$\Rightarrow$

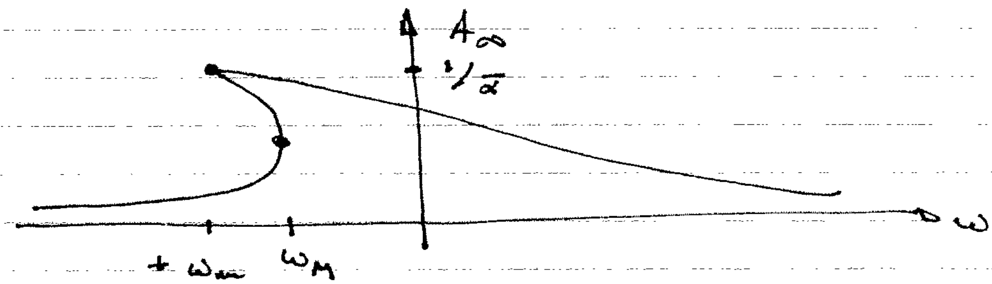
$$\bar{\alpha} A_\infty = -\sin\theta_\infty$$
$$2\omega A_\infty + A_\infty^3 = -\cos\theta_\infty$$

$\Rightarrow$

$$(\bar{\alpha} A_\infty)^2 + (2\omega A_\infty + A_\infty^3)^2 = 1$$

$\Rightarrow$

$$A_\infty^2\left[\bar{\alpha}^2 + (2\omega + A_\infty^2)^2\right] = 1$$



b) for $\omega_m < \omega < \omega_M$ there are three possible solutions (hence they were correct)

2. $t_1 = t,\ t_2 = \varepsilon t$ & $u \sim u_0 + \varepsilon u_1 + \cdots \Rightarrow$

$$T \cdot \partial_x^2 (u_0 + \varepsilon u_1 + \cdots) = (\partial_1^2 + 2\varepsilon \partial_1\partial_2 + \varepsilon^2 \partial_2^2)(u_0 + \varepsilon u_1 + \cdots)$$

$$T = 1 + \varepsilon \int_0^1 (\partial_x u_0 + \cdots)^2 dx \sim 1 + \varepsilon \int_0^1 (\partial_x u_0)^2 dx$$

$O(1)$ $\partial_x^2 u_0 = \partial_1^2 u_0 \quad \Rightarrow \quad u_0 = A(t_1, t_2) \sin\lambda_m x \qquad \lambda_m = m\pi$

$\Rightarrow$

$$\partial_1^2 A = -\lambda_m^2 A$$

$\Rightarrow$

$$A = a(t_2)\cos[\lambda_m t_1 + \theta(t_2)]$$

$O(\varepsilon)$ $\partial_x^2 u_1 + \frac{1}{2}\lambda_m^2 \underset{①}{A^2} \cdot \partial_x^2 u_0 = \partial_1^2 u_1 + \underset{②}{2\partial_1\partial_2 u_0}$

$$① = \tfrac{1}{2}a^2\cos^2(\lambda_m t_1 + \theta)\cdot(-\lambda_m^2 A)\sin\lambda_m x \cdot \lambda_m^2$$
$$= -\tfrac{1}{2}a^3 \cdot \tfrac{1}{4}\left[3\cos(\lambda_m t_1 + \theta) + \text{stuff}\right]\sin\lambda_m x \cdot \lambda_m^4$$
$$= -\tfrac{3}{8}a^3 \cdot \cos(\lambda_m t_1 + \theta)\cdot \sin\lambda_m x \cdot \lambda_m^4 + \text{stuff}$$

$$② = 2\partial_2\left(-a\sin(\lambda_m t_1+\theta)\right)\sin\lambda_m x\cdot\lambda_m$$
$$= \left(-2a'\sin - 2a\theta'\cos\right)\cdot\sin\lambda_m x\cdot\lambda_m$$

$\sin$: $\quad a' = 0 \qquad\Rightarrow\qquad a = \text{const}$

$\cos$: $\quad -\frac{3}{8}\lambda_m^4 a^3 = -2a\theta'\cdot\lambda_m \qquad \theta' = \frac{3}{16}\lambda_m^3 a^2$

$\Rightarrow$

$$u \sim a\cdot\cos\left(\lambda_m t_1 + \frac{3}{16}\lambda_m^3 a^2 t_2 + \varphi_0\right)\cdot\sin\lambda_m x$$

IC's $\Rightarrow$ $a = 1$ & $\varphi_0 = 0$

3. $\theta_1 = x - t,\ \theta_2 = x + t,\ t_2 = \varepsilon t$ & $x_2 = \varepsilon x \quad\Rightarrow$

$$\partial_x \to \partial_{\theta_1} + \partial_{\theta_2} + \varepsilon\partial_{x_2} \quad \& \quad \partial_t \to -\partial_{\theta_1} + \partial_{\theta_2} + \varepsilon\partial_{t_2}$$

$$\partial_x^2 - \partial_t^2 \to 4\partial_{\theta_1}\partial_{\theta_2} + 2\varepsilon\left[(\partial_{\theta_1}+\partial_{\theta_2})\partial_{x_2} + (-\partial_{\theta_1}+\partial_{\theta_2})\partial_{t_2}\right] + \cdots$$

So, $\quad p \sim p_0(\theta_1, \theta_2, x_2, t_2) + \varepsilon p_1 + \cdots \quad\Rightarrow$

$O(1)$ $\quad \partial_{\theta_1}\partial_{\theta_2} p_0 = 0 \quad\Rightarrow\quad p_0 = f_0(\theta_1, x_2) + g_0(\theta_2, x_2)$

note: we're going to assume indep of sol. on t_2

$O(\varepsilon)$ $\quad 4\partial_{\theta_1}\partial_{\theta_2} p_1 = -2\partial_{\theta_1}\partial_{x_2} f_0 - 2\partial_{\theta_2}\partial_{x_2} g_0 - a(x_2)(\partial_{\theta_1} f_0 + \partial_{\theta_2} g)$

$\Rightarrow$

$$4p_1 = -2\theta_2\cdot\partial_{x_2} f_0 - 2\theta_1\partial_{x_2} g_0 - a(x_2)\theta_2 f_0 - a(x_2)\theta_1 g_0$$
$$+ f_1(\theta_1, x_2) + g_1(\theta_2, x_2)$$

$\Rightarrow$

$$-2\partial_{x_2} f_0 - a f_0 = 0 \qquad\Rightarrow\qquad f_0 = F(\theta_1)/\sqrt{A(x_2)}$$
$$-2\partial_{x_2} g_0 - a g_0 = 0 \qquad\qquad g_0 = G(\theta_2)/\sqrt{A(x_2)}$$

$$\therefore\quad p \sim \frac{1}{\sqrt{A(\varepsilon x)}}\left[F(x-t) + G(x+t)\right]$$