Perturbation Methods
HW 4
Due March 5

In the following exercises, when asked to find a first-term expansion of the solution that is valid for large t, you should introduce a time-scale that removes the first secular term that would appear in a regular expansion.

1. Consider the weakly nonlinear oscillator problem of solving

$$y'' + y + \varepsilon y^n = 0, \quad \text{for } 0 < t,$$

where $y(0) = a$ and $y'(0) = 0$. One of the objectives of this exercise is to show that even-order nonlinearities take longer to produce secular terms than those of odd-order.

(a) If $n = 3$, then a secular term appears in the second term of a regular expansion (you do not need to show this). Find a first-term expansion of the solution that is valid for large t.

(b) Taking $n = 2$, and using a regular expansion, show that a secular term does not appear in the second term but does in the third term.

(c) Taking $n = 2$, find a first-term expansion of the solution that is valid for large t.

2. Consider the problem of solving

$$y' = y - y^3, \quad \text{for } 0 < t,$$

where $y(0) = \varepsilon$.

(a) Sketch the direction field for this problem and from this determine what happens to the solution as $t \to \infty$.

(b) Suppose one assumes a regular expansion of the form $y \sim \varepsilon y_0(t) + \varepsilon^\alpha y_1(t) + \cdots$. After finding y_0 and y_1, explain why y_0 is not expected to be an accurate approximation as $t \to \infty$. Also, explain why a multiple-scale expansion should be used but (3.9) will not work for this problem.

(c) Suppose $t_2 = \varepsilon^\alpha f(t)$, where $f(t)$ is determined from the secular removing condition and the requirement that $f(0) = 0$. Show that

$$y \sim \frac{\varepsilon e^t}{\sqrt{1 + \varepsilon^2(e^{2t} - 1)}}.$$

3. 3.18(a)-(c)