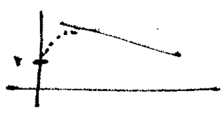
Solutions to HW3

1. a) $\varepsilon y'' + y(y'+3) = 0 \qquad 0<x<1$

$y(0)=1 \qquad y(1)=1$

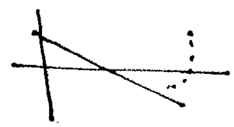
$y_0(1)=1$

outer: $y_0\cdot(y_0'+3)=0 \Rightarrow y_0=0$ or $y_0=-3x+a = 4-3x$



BL $x=0$

$y''<0$
$y(y'+3)>0$ ✓



BL $x=1$

$y''>0$
$y(y'+3)$ $\pm$
✗

BL at $x=0$: $\bar{x} = \frac{x}{\varepsilon} \Rightarrow Y'' + Y(Y'+3\varepsilon)=0$ & $Y(0)=1$

$Y \sim Y_0 \Rightarrow Y_0'' + Y_0Y_0' = 0 \Rightarrow Y_0' + \frac{1}{2}Y_0^2 = A$

$Y_0(\infty) = y_0(0) = 4$ & $Y_0'(\infty)=0 \Rightarrow A = 16$

$\Rightarrow Y_0' = \frac{1}{2}(Y_0^2 - 16) = \frac{1}{2}(Y_0-4)(Y_0+4)$

$$Y_0 = 4\cdot\frac{Be^{4\bar{x}}-1}{Be^{4\bar{x}}+1} \quad \& \quad Y_0(0)=1$$

$$= 4\,\frac{5e^{4x/\varepsilon}-3}{5e^{4x/\varepsilon}+3}$$

$\Rightarrow$

composite: $$y \sim -3x + 4\,\frac{5e^{4x/3}-3}{5e^{4x/3}+3}$$

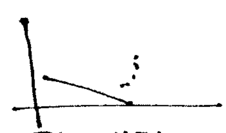
b) $\varepsilon y'' - y(y'+1) = 0$

$y(0) = y(1) = 3$

outer: $y \sim y_0 + \dots \Rightarrow y_0\cdot(y_0'+1)=0 \Rightarrow y_0 = 0$ or $y_0 = -x+a$
$= 3-x$

BL $x=0$

$y''<0$
$y(y'+1)>0$
✗



BL $x=1$

$y''>0$
$y(y'+1)>0$ ✓

BL at $x=1$: $\bar{x} = \frac{x-1}{\varepsilon} \Rightarrow x = 1+\varepsilon\bar{x} \Rightarrow Y'' - Y(Y'+\varepsilon)=0$

$Y \sim Y_0 + \dots \Rightarrow Y_0'' - Y_0Y_0' = 0 \Rightarrow Y_0' - Y_0^2\cdot\frac{1}{2} = A$

$Y_0(-\infty) = y_0(1) = 2$ & $Y'(-\infty)=0 \Rightarrow A = -2$

$Y_0' = -2 + \frac{1}{2}Y_0^2 = -2(4-Y_0^2) = -2(2-Y_0)(2+Y_0)$

$$Y_{10} = 2\,\frac{3e^{-2\bar{x}}-1}{3e^{-2\bar{x}}+1} \qquad Y_0(0)=3$$

$$= 2\,\frac{5e^{-2\bar{x}}+1}{5e^{-2\bar{x}}-1}$$

composite:

$$y \sim 1-x+2\cdot\frac{5e^{-2\cdot\frac{x-1}{\varepsilon}}+1}{5e^{-2\cdot\frac{x-1}{\varepsilon}}-1}$$

2. a) $\varepsilon\partial_x(H^3yy') = \partial_x(Hy) \qquad H(0)\neq H(1)$

$y(0)=y(1)=1$

outer: $y\sim y_0+\dots \Rightarrow \partial_x(Hy_0)=0 \Rightarrow Hy_0=\text{const} \qquad y_0(0)=1$

$$y_0=\frac{H(0)}{H(x)}$$

BL at $x=1$: $\bar{x}=\frac{x-1}{\varepsilon} \Rightarrow x=1+\varepsilon\bar{x} \Rightarrow$

$$\partial_{\bar{x}}\left[H^3(1+\varepsilon\bar{x})\,Y\,Y'\right] = \partial_{\bar{x}}\left[H(1+\varepsilon\bar{x})\,Y\right]$$

$Y\sim Y_0+\dots \Rightarrow \quad \partial_{\bar{x}}\left[H^3(1)\,Y_{10}Y_{10}'\right]=\partial_{\bar{x}}\left[H(1)\,Y_{10}\right]$

$\Rightarrow$

$$H_1^2\,Y_0Y_0' = Y_0 + A$$

$Y_0(-\infty)=H_0/H_1$ & $Y_0'(-\infty)=0 \Rightarrow A=-\frac{H_0}{H_1}$

$\Rightarrow$

$$H_1^2\,Y_0Y_0' = Y_0-\frac{H_0}{H_1} \Rightarrow \frac{Y_0}{Y_0-H_0/H_1}\cdot H_1\,dY_0 = d\bar{x}$$

$\Rightarrow$

$$H_1^2\left[Y_0+\frac{H_0}{H_1}\ln\left(Y_0-\frac{H_0}{H_1}\right)\right] = \bar{x}+B \qquad Y_0(0)=1$$

$$B = H_1^2\left[1+\frac{H_0}{H_1}\ln\left(1-\frac{H_0}{H_1}\right)\right]$$

$\Rightarrow$

$$\bar{x} = H_1^2(Y_0-1)+H_0H_1\cdot\ln\left|\frac{H_1Y_0-H_0}{H_1-H_0}\right|$$

composite: $y\sim y_0+Y_0-y_0(1)$

$$=\frac{H_0}{H}+Y_{10}-\frac{H_0}{H_1}$$

3. $\varepsilon y'' = y y' - y^3$

$y(0) = \frac{3}{5} \qquad y(1) = -\frac{2}{3}$

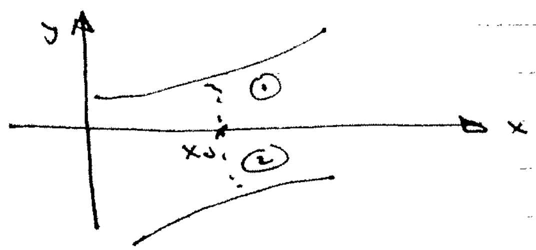
outer: $y_0 y_0' - y_0^3 = 0 \Rightarrow y_0 = 0$ or $y_0' = y_0^2 \Rightarrow \frac{dy_0}{y_0^2} = dx$

$\Rightarrow -\frac{1}{y_0} = x + a \Rightarrow y_0 = \frac{-1}{x+a}$

$y_0(0) = \frac{3}{5} \Rightarrow a = -\frac{5}{3} \Rightarrow y_0 = \frac{3}{5-3x}$

$y_0(1) = -\frac{2}{3} \Rightarrow a = \frac{1}{2} \Rightarrow y_0 = \frac{-2}{2x+1}$

$\therefore y_0 = \begin{cases} \frac{3}{5-3x} & 0 \le x < x_0 \\ \frac{-2}{2x+1} & x_0 < x \le 1 \end{cases}$



① $y'' < 0$
$y(y' - y^2) < 0$ ✓
$y > 0 \Rightarrow y(x_0) = 0$

② $y'' > 0$
$y(y' - y^2) > 0$ ✓

IL at $x = x_0$: $\bar{x} = \frac{x - x_0}{\varepsilon} \Rightarrow Y'' = YY' - \varepsilon Y^3$

$Y \sim Y_0 + \cdots \Rightarrow Y_0'' = Y_0 Y_0' \Rightarrow Y_0' = \frac{1}{2} Y_0^2 + A$

$Y_0'(\pm\infty) = 0$, $Y_0(-\infty) = y_0(x_0^+)$ & $Y_0(+\infty) = y_0(x_0^-) \Rightarrow$

$$0 = \frac{1}{2}\left(\frac{3}{5 - 3x_0}\right)^2 + A \qquad x_0 = \frac{7}{12}$$
$$0 = \frac{1}{2}\left(\frac{-2}{2x_0 + 1}\right)^2 + A \Rightarrow A = -\frac{1}{2}\left(\frac{12}{13}\right)^2 = -\frac{1}{2}\alpha^2$$

$\Rightarrow \frac{dY_0}{Y_0^2 - \alpha^2} = \frac{1}{2} d\bar{x}$

$\Rightarrow$

$Y_0 = \alpha \tanh\left(-\frac{1}{2}\alpha \bar{x} + B\right) \qquad Y_0(0) = 0 \Rightarrow B = \frac{1}{2}\alpha$

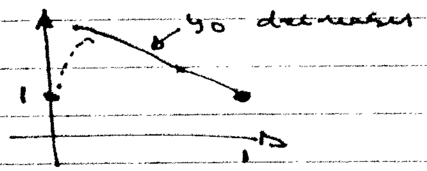
$= \alpha \tanh \frac{1}{2}\alpha(1 - \bar{x})$

4. $\varepsilon y'' = p(y) y' + q(y)$

$y(0) = y(1) = 1$

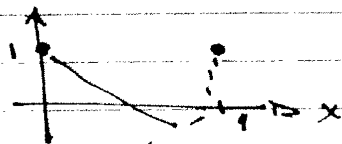
outer: $y \sim y_0 + \cdots \Rightarrow y_0' = -\frac{q(y_0)}{p(y_0)}$

a) $p = e^y$ & $q = 5 + y^2 \Rightarrow \quad y_0' < 0 \ \forall x$



$y'' < 0$
$py' + q > 0$ $\times$ not possible

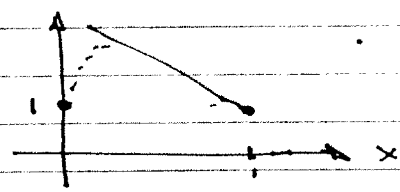
BL at $x = 0$



$y'' > 0$
$py' + q > 0$ ✓ $\therefore$ this is possible

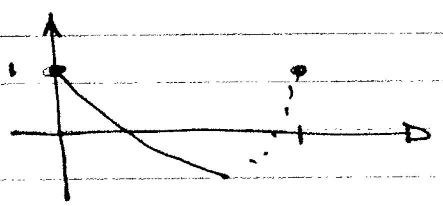
BL at $x = 1$

b) $p = -e^y$ & $q = -(1 + y^2) \Rightarrow \quad y_0' < 0 \ \forall x$



$y'' < 0$
$py' + q < 0$ ✓ $\therefore$ this is possible

BL at $x = 0$



$y'' > 0$
$py' + q < 0$ $\times$ not possible

BL at $x = 1$