1.a) $y'' + \varepsilon y' - y = 1$ $\quad$ $y \sim y_0 + \varepsilon y_1$

$y(0) = 1 \quad y(1) = 0$

$O(1)$ $y_0'' = y_0 + 1 \quad \Rightarrow \quad y_0 = -1 + Ae^{x} + Be^{-x}$

$y_0(0) = 1 \quad y_0(1) = 0$

$-1 + A + B = 1$

$-1 + Ae + Be^{-} = 0$

$B = 2 - A$

$-1 + Ae + e^{-}(2-A) = 0 \qquad A(e - e^{-}) = 1 - 2e^{-}$

$A = \dfrac{1-2e^{-}}{e-e^{-}} \qquad B = 2 - \dfrac{1-2e^{-}}{e-e^{-}} = \dfrac{2e - 2e^{-} - 1 + 2e^{-}}{e - e^{-}}$

$= \dfrac{2e-1}{e-e^{-}}$

$O(\varepsilon)$ $y_1'' - y_1 = -y_0' = Ae^{x} - Be^{-x}$

$y_1 = axe^{x} + bxe^{-x}$

$y_1' = a(xe^{x} + e^{x}) + b(-xe^{-x} + e^{-x})$

$y_1'' = a(2e^{x} + xe^{x}) + b(-2e^{-x} + xe^{-x})$

$\Rightarrow$

$a(2e^{x} + \cancel{xe^{x}}) + b(-2e^{-x} + \cancel{xe^{-x}}) - (\cancel{axe^{x}} + \cancel{bxe^{-x}}) = Ae^{x} - Be^{-x}$

$\Rightarrow$

$2a = A \quad \Rightarrow \quad a = \frac{1}{2}A$

$-2b = -B \qquad b = \frac{1}{2}B$

$\Rightarrow y_1 = axe^{x} + bxe^{-x} + \alpha e^{x} + \beta e^{-x}$

$\alpha + \beta = 0$

$ae + be^{-} + \alpha e + \beta e^{-} = 0$

$\Rightarrow$

$\beta = -\alpha$

$\alpha(e - e^{-}) = -ae - be^{-}$

$= -\frac{1}{2}Ae - \frac{1}{2}Be^{-} = -\frac{1}{2}\dfrac{1-2e^{-}}{e-e^{-}} \cdot e - \frac{1}{2}\dfrac{2e-1}{e-e^{-}}e^{-}$

$= -\frac{1}{2}\dfrac{e-2}{e-e^{-}} - \frac{1}{2}\dfrac{2-e^{-}}{e-e^{-}}$

$= -\frac{1}{2}\dfrac{1}{e-e^{-}}(e - 2 + 2 - e^{-}) = -\frac{1}{2}$

$$\therefore \quad y_1 = axe^{x} + bxe^{-x}\left[-\tfrac{1}{2}e^{x} + \tfrac{1}{2}e^{-x}\right]\cdot\frac{1}{e-e^{-1}}$$

$$a = \frac{1}{2}\,\frac{1-2e^{-}}{e-e^{-}} \qquad b = \frac{1}{2}\,\frac{2e-1}{e-e^{-}}$$

$$y_0 = -1 + Ae^{x} + Be^{-x}$$

$$= -1 + \frac{1-2e^{-}}{e-e^{-}}\,e^{x} + \frac{2e-1}{e-e^{-}}\,e^{-x}$$

$$= -1 + \frac{1}{e-e^{-}}\left[e^{x} - 2e^{x-1} + 2e^{1-x} - e^{-x}\right]$$

$$2\sinh(x) + 2\cdot 2\sinh(1-x)$$

$$= -1 + \frac{\sinh(x) + 2\sinh(1-x)}{\sinh(1)}$$

$x=0$: $y_0 = -1 + 2 = 1$

$x=1$: $y_0 = -1 + \frac{\sinh(1)}{\sinh(1)} = 0$

$$y_1 = \frac{1}{2}\,\frac{1-2e^{-}}{e-e^{-}}\,xe^{x} + \frac{1}{2}\,\frac{2e-1}{e-e^{-}}\,xe^{-x} \;\frac{-\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}}{e-e^{-1}}$$

$$= \frac{1}{2}\,\frac{1}{e-e^{-1}}\left[xe^{x} - 2xe^{-1+x} + 2xe^{1-x} - xe^{-x}\right] - \frac{1}{2}\,\frac{(e^{x} - e^{-x})}{e-e^{-1}}$$

$$= \frac{x}{2(e-e^{-1})}\left[e^{x} - 2e^{-1+x} + 2e^{1-x} - e^{-x}\right] - \frac{\sinh(x)}{2\sinh(1)}$$

$$2\sinh(x) + 2\cdot 2\sinh(1-x)$$

$$= \frac{2x\left[\sinh(x) + 2\sinh(1-x)\right]}{4\cdot\sinh(1)} - \sinh(x)$$

$$= \frac{x\left[\sinh(x) + 2\sinh(1-x)\right]}{2\sinh(1)} - \frac{\sinh(x)}{2\sinh(1)}$$

$x=0$: $y_1(0) = 0$

$x=1$ $\quad y_1 = \frac{\sinh}{2}$

$$y_1 = \frac{(x-1)\sinh(x) + 2x\sinh(1-x)}{2\sinh(1)}$$

1 b) $y'' - y + \varepsilon y^3 = 0$ $\qquad y \sim y_0 + \varepsilon y_1$

$y(0) = 0 \qquad y(1) = 1$

O(1) $y_0'' = y_0 \quad \Rightarrow \quad y_0 = \cancel{A \sin x + B \cos x}$ $\; Ae^x + Be^{-x}$

$y_0(0) = 0 \qquad y_0(1) = 0$

$y_0(0) = 0 \Rightarrow A + B = 0 \qquad \Rightarrow \qquad B = -A$

$y_0(1) = 1 \Rightarrow Ae + Be^{-1} = 1 \qquad A(e - e^{-1}) = 1$

O(2) $y_1'' = y_1 - y_0^3$

$y_1(0) = y_1(1) = 0 \qquad y_0 = \frac{e^x - e^{-x}}{e - e^{-1}} = \frac{\sinh(x)}{\sinh(1)}$

$y_0^3 = (Ae^x + Be^{-x})^3 \qquad \Delta = \sinh(1)$

$$= \frac{\sinh^3(x)}{\Delta^3}$$

$$= \frac{1}{\Delta^3} \cdot \frac{1}{4}\left[\sinh(3x) - 3\sinh(x)\right]$$

$$y_1 = a \sinh 3x + b \cosh 3x + c x \sinh x + d x \cosh x$$

$$y_1' = 3a \cosh 3x + 3b \sinh 3x + c(\sinh x + x \cosh x) + d(\cosh x + x \sinh x)$$

$$y_1'' = 9a \sinh 3x + 9b \cosh 3x + c(2\cosh x + x \sinh x) + d(2\sinh x + x \cosh x)$$

$\Rightarrow$

$$9a \underline{\sinh 3x} + 9b \cosh 3x + c(2\cosh x + \cancel{x \sinh x}) + d(2\sinh x + \cancel{x \cosh x})$$

$$= a \underline{\sinh 3x} + b \cosh 3x + \cancel{c x \sinh x} + \cancel{d x \cosh x} - \frac{1}{4\Delta^3}\left[\underline{\sinh 3x} - 3\sinh x\right]$$

$\Rightarrow$

$9a = a - \frac{1}{4\Delta^3} \quad \Rightarrow \quad a = -\frac{1}{32\Delta^3}$

$b = 0 \qquad c = 0$

$2d = \frac{3}{4\Delta^3} \qquad d = \frac{3}{8\Delta^3}$

$$y_1 = a\sinh 3x + dx\cosh x + \alpha\sinh x + \beta\cosh x$$

$y_1(0)=0 \Rightarrow \beta = 0$

$y_1(1)=0 \Rightarrow \quad a\sinh 3 + d\cosh(1) + \alpha\sinh 1 = 0$

$$\alpha = -\frac{a\sinh 3 + d\cosh 1}{\sinh 1}$$

$$a = -\frac{1}{32}\cdot\frac{1}{\sinh^3(1)} \qquad d = \frac{3}{8\sinh^3(1)}$$

$$\therefore \quad y_0 = \frac{\sinh(x)}{\sinh(1)}$$

$$y_1 = a\sinh 3x + dx\cosh x + \alpha\sinh x$$

c) $y'' - y + y^3 = 0$ $\qquad y \sim \varepsilon y_0 + \varepsilon^\alpha y_1 \qquad \alpha = 3$

$y(0)=0, \; y(1)=\varepsilon$ $\qquad y^3 \sim \varepsilon^3 y_0^3 + \cdots$

$O(\varepsilon)$ $\quad y_0'' - y_0 = 0$ $\qquad y_0$ same as in part (b)

$y_0(0)=0, \; y_0(1)=1$

$O(\varepsilon^3)$ $\quad y_1'' = y_1 - y_0^3$ $\qquad y_1$ same as in part (b)

$y_1(0)=y_1(1)=0$

2. $$y'' + \omega^2\left(\frac{1}{\gamma\sqrt{1+\varepsilon^2 y^2}} + \alpha^2\right)y = 0 \qquad 0<x<1$$
$$y'(0) = y'(1) = 0 \qquad 0<\alpha<1$$

a) $$y \sim y_0 + \varepsilon^2 y_1 + \cdots$$
$\Rightarrow$
$$\sqrt{1+\varepsilon^2 y^2} \sim 1 + \tfrac{1}{2}y^2\varepsilon^2 \sim 1 + \tfrac{1}{2}\varepsilon^2 y_0^2$$
$$\frac{1}{\sqrt{1+\varepsilon^2 y^2}} \sim 1 - \tfrac{1}{2}\varepsilon^2 y_0^2$$
$$\gamma \approx \frac{1}{\alpha^2}\Big[\underbrace{\frac{1}{1-\alpha^2} - 1}_{\frac{1-1+\alpha^2}{1-\alpha^2}} + \tfrac{1}{2}\varepsilon^2\int_0^1 y_0^2\,dx\Big] \qquad \gamma_0 = \frac{1}{1-\alpha^2}$$
$$= \frac{1}{1-\alpha^2} + \frac{1}{2\alpha^2}\varepsilon^2\int_0^1 y_0^2\,dx = \gamma_0 + \varepsilon^2\gamma_1 \qquad \gamma_1 = \frac{1}{2\alpha^2}\int_0^1 y_0^2 dx$$
$$\frac{1}{\gamma\sqrt{1+\varepsilon^2 y^2}} \sim \frac{1}{\gamma_0+\varepsilon^2\gamma_1}\cdot\frac{1}{1+\tfrac{1}{2}\varepsilon^2 y_0^2}$$
$$\sim \frac{1}{\gamma_0}\cdot\left(1 - \varepsilon^2\frac{\gamma_1}{\gamma_0}\right)\left(1 - \tfrac{1}{2}\varepsilon^2 y_0^2\right)$$
$$= \frac{1}{\gamma_0}\left[1 + \varepsilon^2\left(-\frac{\gamma_1}{\gamma_0} - \tfrac{1}{2}y_0^2\right)\right]$$

$$\omega \sim \omega_0 + \varepsilon^2\omega_1 \Rightarrow \omega^2 \sim \omega_0^2 + 2\varepsilon^2\omega_0\omega_1$$
$$\omega^2\cdot\left[\alpha^2 + \frac{1}{\gamma\sqrt{1+\varepsilon^2 y^2}}\right]$$
$$\sim (\omega_0^2 + 2\varepsilon^2\omega_0\omega_1)\left[\alpha^2 + \frac{1}{\gamma_0}\left[1 + \varepsilon^2\left(-\frac{\gamma_1}{\gamma_0} - \tfrac{1}{2}y_0^2\right)\right]\right]$$
$$\underbrace{\alpha^2 + \frac{1}{\gamma_0}}_{=1} - \frac{1}{\gamma_0}\varepsilon^2\left(\frac{\gamma_1}{\gamma_0} + \tfrac{1}{2}y_0^2\right)$$
$$= \omega_0^2 + \varepsilon^2\left[-\frac{\omega_0^2}{\gamma_0}\left(\frac{\gamma_1}{\gamma_0} + \tfrac{1}{2}y_0^2\right) + 2\omega_0\omega_1\right].$$
$$y_0'' + \varepsilon^2 y_1'' + \cdots + \left\{\omega_0^2 + \varepsilon^2\left[2\omega_0\omega_1 - \frac{\omega_0^2}{\gamma_0}\left(\frac{\gamma_1}{\gamma_0} + \tfrac{1}{2}y_0^2\right)\right]\right\}(y_0 + \varepsilon^2 y_1 + \cdots$$

$O(1)$ $$y_0'' + \omega_0^2 y_0 = 0 \quad\Rightarrow\quad y_0 = A\sin\omega_0 x + B\cos\omega_0 x$$
$$y_0'(0) = y_0'(1) = 0 \qquad y_0' = \omega_0 A\cos\omega_0 x - \omega_0 B\sin\omega_0 x$$

$$y_0'(0)=0 \Rightarrow A=0$$
$$y_0'(1)=0 \Rightarrow \sin\omega_0 = 0 \qquad \Rightarrow \omega_0 = n\pi \qquad n\in\mathbb{Z}$$

$O(\varepsilon^2)$ $\quad y_1'' + \omega_0^2 y_1 = -y_0\cdot\left[2\omega_0\omega_1 - \frac{\omega_0^2}{\gamma_0}\cdot\left(\frac{\gamma_1}{\gamma_0} + \frac{1}{2}y_0^2\right)\right]$

$y_0 = B\cos\omega_0 x$

$$\int_0^1 y_0^2\,dx = B^2\int_0^1 \cos^2\omega_0 x\,dx = B^2\left(\frac{1}{2}x + \frac{1}{4\omega_0}\sin 2\omega_0 x\right)\Big|_{x=0}^{1} = \frac{1}{2}B^2$$

$$\int_0^1 y_0^4\,dx = B^4\left(\frac{3x}{8} + \frac{\sin 2\omega_0 x}{4\omega_0} + \frac{\sin 4\omega_0 x}{32\omega_0}\right)\Big|_{x=0}^{1} = \frac{3}{8}B^4$$

$$2\omega_0\omega_1\int_0^1 y_0^2\,dx = \frac{\omega_0^2}{\gamma_0}\left[\frac{\gamma_1}{\gamma_0}\int_0^1 y_0^2\,dx + \frac{1}{2}\int_0^1 y_0^4\,dx\right]$$

$$\frac{1}{2}B^2\cdot 2\omega_0\omega_1 = \frac{\omega_0^2}{\gamma_0}\left[\frac{\gamma_1}{\gamma_0}\cdot\frac{1}{2}B^2 + \frac{1}{2}\cdot\frac{3}{8}B^4\right]$$

$$\gamma_1 = \frac{1}{2\alpha^2}\int_0^1 y_0^2\,dx = \frac{1}{2\alpha^2}\cdot\frac{1}{2}B^2$$

$\Rightarrow$

$$\omega_0\omega_1 = \frac{\omega_0^2}{\gamma_0}\left[\frac{1}{2}\frac{1}{\gamma_0}\cdot\frac{1}{4\alpha^2} + \frac{3}{16}\right]B^2$$

$\Rightarrow$

$$\omega_1 = \omega_0(1-\alpha^2)\left[\frac{1}{8}\frac{1-\alpha^2}{\alpha^2} + \frac{3}{16}\right]B^2$$

$$= \omega_0(1-\alpha^2)\cdot\frac{1}{16\alpha^2}\left[2(1-\alpha^2) + 3\alpha^2\right]B^2$$

$$= \omega_0(1-\alpha^2)\frac{1}{16\alpha^2}(2+\alpha^2)B^2$$

$\Rightarrow$

$$\omega \sim \omega_0\cdot\left[1 + \varepsilon^2\cdot\frac{1}{16\alpha^2}(1-\alpha^2)(2+\alpha^2)B^2\right]$$

b) Note that γ is not defined at $\alpha=0$ or at $\alpha=1$. There are no apparent problems in the expansion for ω as $\alpha\uparrow 1$ but there is a problem as $\alpha\downarrow 0$ (basically the expansion becomes not well ordered).

c) Since $\omega_1 > 0$ and $\omega_0 = n\pi$ (so, $n=1$ is smallest such ω_0) then all frequencies are greater than π.

3. a) $k=2,\ z_1=1,\ z_2=-1$

$$\nabla^2\phi = -\sum_{i=1}^{k}\alpha_i z_i e^{-z_i\phi}$$

$$= -\left(\alpha_1 z_1 e^{-z_1\phi} + \alpha_2 z_2 e^{-z_2\phi}\right)$$

$$= -\left(\alpha_1 e^{-\phi} - \alpha_2 e^{\phi}\right)$$

$\alpha_1 z_1 + \alpha_2 z_2 = 0 \Rightarrow \alpha_1 - \alpha_2 = 0,\quad \alpha_1 = \alpha_2$

$$= +\alpha_1\left(e^{\phi} - e^{-\phi}\right)$$

$$= 2\alpha_1 \sinh(\phi)$$

$$\partial_n\phi\big|_{\partial\Omega} = -\varepsilon$$

$$\phi \sim \varepsilon\phi_0 + \varepsilon^\alpha\phi_1 + \cdots$$

$$\sinh(\varepsilon\phi_0 + \varepsilon^\alpha\phi_1 + \cdots)$$
$$= \varepsilon\phi_0 + \varepsilon^\alpha\phi_1 + \cdots + \tfrac{1}{6}(\varepsilon\phi_0 + \cdots)^3 \quad \Rightarrow \alpha = 3$$
$$= \varepsilon\phi_0 + \varepsilon^3\left(\phi_1 + \tfrac{1}{6}\phi_0^3\right)$$

$O(\varepsilon)$: $\nabla^2\phi_0 = \alpha\phi_0$

$\partial_n\phi_0\big|_{\partial\Omega} = 1$

$O(\varepsilon^3)$: $\nabla^2\phi_1 = \phi_1 + \frac{1}{6}\phi_0^3$

$\partial_n\phi_1\big|_{\partial\Omega} = 0$