

Perturbation Methods
HW 2
Due Feb 9

1. Find a two term asymptotic expansion, for small ε , of the solution of the following problems:

- (a) $y'' + \varepsilon y' - y = 1$, where $y(0) = 1$ and $y(1) = 0$.
- (b) $y'' - y + \varepsilon y^3 = 0$, where $y(0) = 0$ and $y(1) = 1$.
- (c) $y'' - y + y^3 = 0$, where $y(0) = 0$ and $y(1) = \varepsilon$.

2. An interesting problem that can trace its origins back to Bernoulli and Euler is that of a whirling elastic string that is held fixed at each end (something like a jump rope). The problem comes down to solving for a function $y(x)$ and a frequency of rotation ω that satisfy [Caughey 1970]:

$$y'' + \omega^2 \left(\frac{1}{\gamma \sqrt{1 + \varepsilon^2 y^2}} + \alpha^2 \right) y = 0, \quad \text{for } 0 < x < 1,$$

where $y'(0) = y'(1) = 0$. Here α is a constant that satisfies $0 < \alpha < 1$, and γ is a constant that depends on y through the equation

$$\gamma = \frac{1}{\alpha^2} \left(\frac{1}{1 - \alpha^2} - \int_0^1 \frac{dx}{\sqrt{1 + \varepsilon^2 y^2}} \right).$$

There is one additional constraint imposed on the solution, and it is that

$$\int_0^1 y(x) dx = 0.$$

What we have here is an eigenvalue problem where ω is the eigenvalue.

- (a) Find a two term expansion of ω for small ε . Note that to do this you will need to find the first term in the expansion for $y(x)$, but not necessarily the second term.
- (b) Comment on the accuracy of the expansion for ω for $0 < \alpha < 1$.
- (c) Caughey [1970] proves that whirling can not occur at frequencies below $\omega = \pi$. Is your expansion in part (a) consistent with this result?

3. In the nonlinear potential problem example in Section 1.6, suppose that $k = 2$, $z_1 = 1$ and $z_2 = -1$.

- (a) Show that the equation takes the form

$$\nabla^2 \phi = \alpha \sinh(\phi), \quad \text{for } \mathbf{x} \in \Omega,$$

where α is a positive constant and the boundary condition is given in (1.54).

- (b) Find the problems satisfied by the first and second terms in the expansion for ϕ (you do not need to solve these problems).