Solutions to HW1

1.2 a) $f = O(\varepsilon^\alpha) \Rightarrow \exists k_0, \varepsilon_1 \ni |f| \le k_0 \varepsilon^\alpha = k_0 \varepsilon^{\alpha-\beta} \cdot \varepsilon^\beta$
so, given $\delta > 0$ take $\varepsilon_2 \ni k_0 \varepsilon_2^{\alpha-\beta} < \delta \Rightarrow$
$|f| \le \delta \cdot \varepsilon^\beta \Rightarrow f = o(\varepsilon^\beta)$

b) $f = O(g) \Rightarrow \exists k_0, \varepsilon_1 \ni |f| \le k_0 |g| \Rightarrow |f|^\gamma \le k_0^\gamma |g|^\gamma \Rightarrow |f|^\gamma = O(|g|^\gamma)$
Note, $\varepsilon^2 = O(\varepsilon)$ but $(\varepsilon^2)^{-1} \neq O(\varepsilon^{-1})$

c) take (e.g.) $f = \frac{1}{\varepsilon}$ & $g = -\frac{1}{\varepsilon}$

1.6 d) $\phi_1 \ll \phi_5 \ll \phi_2 \ll \phi_3 \gg \phi_6 \gg \phi_4$

pf: $\frac{\phi_1}{\phi_5} = \frac{\varepsilon^5 e^{-3/\varepsilon}}{\frac{1}{2}\varepsilon^3 \sin \varepsilon^3} = \frac{\varepsilon^5 e^{-3/\varepsilon}}{\frac{1}{2} \cdot O(\varepsilon^3)} = O(\varepsilon^3) \cdot e^{-\frac{3}{\varepsilon}} \to 0$

$\frac{\phi_1}{\phi_2} = \frac{O(\varepsilon^2)}{\varepsilon} = O(\varepsilon) \to 0$

$\frac{\phi_2}{\phi_3} = \frac{\varepsilon}{\varepsilon \ln \varepsilon} = \frac{1}{\ln \varepsilon} \to 0$

$\frac{\phi_3}{\phi_6} = \frac{\varepsilon \ln \varepsilon}{3 \ln \frac{1}{\varepsilon}} = \varepsilon \ln^2 \varepsilon \to 0$ (l'Hos: $\varepsilon \ln^2 \varepsilon \to \frac{\ln^2 \varepsilon}{1/\varepsilon} \to \frac{2 \ln \varepsilon \cdot \frac{1}{\varepsilon}}{-1/\varepsilon^2} = -2\varepsilon \ln \varepsilon \to 0$)

$\frac{\phi_6}{\phi_4} = \frac{3 \ln \frac{1}{\varepsilon}}{e^{-\frac{3}{\varepsilon}}} = 3 \ln \varepsilon \cdot e^{\frac{3}{\varepsilon}} \to 0$

1.7 b) $\left(1 + \frac{1}{\cos \varepsilon}\right)^{3/2} \sim \left(1 + \frac{1}{1 - \frac{1}{2}\varepsilon^2 + \cdots}\right)^{3/2}$

$\sim \left(1 + 1 + \frac{1}{2}\varepsilon^2 + \cdots\right)^{3/2} = \left(2 + \frac{1}{2}\varepsilon^2 + \cdots\right)^{3/2}$

$\sim 2^{3/2} + \frac{3}{4}\sqrt{2}\,\varepsilon^2$

i) $\int_0^\pi \frac{\sin x \, dx}{\sqrt{1 + \varepsilon x}} \sim \int_0^\pi \sin x \cdot \left(1 - \frac{1}{2}\varepsilon x + \cdots\right) dx$

$= 2 - \frac{1}{2}\varepsilon\pi + \cdots$

1.17 h) $x^2 + \varepsilon\sqrt{2 + x} = \cos \varepsilon$

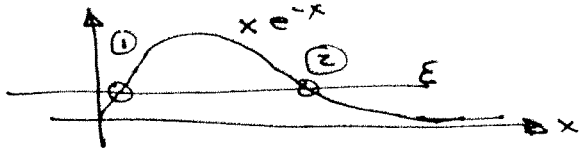
$x \sim x_0 + \varepsilon^\alpha x_1 + \cdots \Rightarrow \underline{x_0^2} + \underline{\underline{2\varepsilon^\alpha x_0 x_1 + \cdots}} + \underline{\underline{\varepsilon\sqrt{2 + x_0 + \cdots}}} = \underline{1 - \frac{1}{2}\varepsilon^2 + \cdots}$

$O(1)$ $x_0^2 = 1 \Rightarrow x_0 = \pm 1$ and $\alpha = 1$

$O(\varepsilon)$ $2x_0 x_1 + \sqrt{\varepsilon + x_0} = 0 \Rightarrow x_1 = -\frac{1}{2x_0}\sqrt{\varepsilon + x_0}$

$\therefore$ $x \sim 1 - \frac{1}{2}\sqrt{3}\,\varepsilon$ and $x \sim -1 + \frac{1}{2}\varepsilon$

b) $x e^{-x} = \varepsilon$



① $x \sim \varepsilon x_0 + \varepsilon^2 x_1 + \cdots \Rightarrow$

$(\varepsilon x_0 + \varepsilon^2 x_1 + \cdots)\cdot[1 - (\varepsilon x_0 + \cdots) + \cdots] = \varepsilon$

$O(\varepsilon)$ $x_0 = 1$ $\qquad$ $O(\varepsilon^2)$ $x_1 - x_0^2 = 0 \Rightarrow x_1 = 1$

$\therefore$ $x \sim \varepsilon + \varepsilon^2$

② $x e^{-x} = \varepsilon \Rightarrow \ln x - x = \ln \varepsilon$

$x \nearrow \infty \Rightarrow \ln x << x \Rightarrow x_0 = -\ln \varepsilon$

$\Rightarrow$ $x \sim -\ln\varepsilon + \mu(\varepsilon)$ $\qquad$ $\mu << \ln\varepsilon$

$\ln(-\ln\varepsilon + \mu) - (-\ln\varepsilon + \mu) = \ln\varepsilon$

$\ln[-\ln\varepsilon(1 + \frac{\mu}{\ln\varepsilon})]$

$= \ln(-\ln\varepsilon) + \ln(1 + \frac{\mu}{\ln\varepsilon})$

$\ln(1) + \frac{\mu}{\ln\varepsilon} + \cdots$

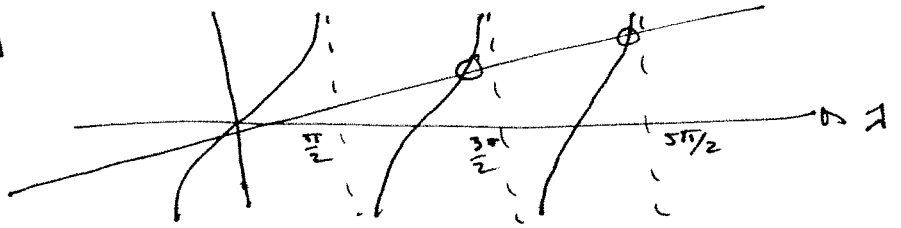
$\therefore$ $\ln(-\ln\varepsilon) - \mu = 0$

$\mu = \ln(-\ln\varepsilon)$

$\Rightarrow$

$x \sim -\ln\varepsilon + \ln(-\ln\varepsilon)$

1.19 a) $\tan\lambda = \lambda$



From periodicity of $\tan\lambda$ the for large λ the solutions are (approx) $\lambda \approx \frac{1}{2}(2n-1)\pi$ for $n \in \mathbb{N}$

Setting $\theta_n = \frac{1}{2}(2n-1)\pi$, then for θ near θ_n

$$\sin\theta \sim 1 - \tfrac{1}{2}(\theta-\theta_n)^2$$
$$\cos\theta \sim -(\theta-\theta_n) + \tfrac{1}{6}(\theta-\theta_n)^3$$

$$\Rightarrow \quad \tan\theta \sim \frac{-1}{\theta-\theta_n}\,\frac{1-\frac{1}{2}(\theta-\theta_n)^2}{1-\frac{1}{6}(\theta-\theta_n)^2} \sim \frac{-1}{\theta-\theta_n}\left[1-\tfrac{1}{3}(\theta-\theta_n)^2\right]$$

Since $\theta \to \theta_n$ as $n \to \infty$, then for large n, $\theta \sim \theta_n + \mu$ (where $\frac{\mu}{\theta_n} \ll 1$) $\Rightarrow$

$$\theta_n + \mu \sim \frac{-1}{\mu}\left(1 - \tfrac{1}{3}\mu^2 + \cdots\right)$$

$$\Rightarrow \quad \theta_n\mu + \mu^2 = -1 + \tfrac{1}{3}\mu^2 + \cdots$$

$$\Rightarrow \quad \mu = -\frac{1}{\theta_n}$$

$$\therefore \quad \theta \sim \theta_n - \frac{1}{\theta_n} = \theta_n\left(1 - \frac{1}{\theta_n^2}\right)$$

$$\varepsilon = \frac{1}{\theta_n} \quad (\alpha = 1) \qquad \beta = 2, \quad \theta_1 = -1$$