

Summary for Stability for Reaction-Diffusion Problems

One Equation

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + f(c)$$

Uniform Steady-state: $f(\bar{c}) = 0$

Stability: A steady-state is asymptotically stable if $f'(\bar{c}) < 0$ and it is unstable if $f'(\bar{c}) > 0$

Two Equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} + f(u, v) \\ \frac{\partial v}{\partial t} &= D_2 \frac{\partial^2 v}{\partial x^2} + g(u, v) \end{aligned}$$

Uniform Steady-state: $f(\bar{u}, \bar{v}) = 0, g(\bar{u}, \bar{v}) = 0$

The Jacobian \mathbf{J} is

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}$$

Also, let

$$K = D_2 \frac{\partial f}{\partial u} + D_1 \frac{\partial g}{\partial v}$$

In the theorem below, \mathbf{J} and K are evaluated at (\bar{u}, \bar{v}) .

Theorem: Assuming that D_1 and D_2 are positive, then a uniform steady-state is asymptotically stable if:

1. $\text{tr}(\mathbf{J}) < 0$

and

2. One of the following hold:

- a) $K \leq 0$ and $\det(\mathbf{J}) > 0$

or

- b) $K > 0$ and $4D_1D_2\det(\mathbf{J}) > K^2$

It is unstable if (i) $\text{tr}(\mathbf{J}) > 0$, or if (ii) $K \leq 0$ and $\det(\mathbf{J}) < 0$, or if (iii) $K > 0$ and $4D_1D_2\det(\mathbf{J}) < K^2$.

Turing Instability: This occurs if:

1. A uniform steady-state is asymptotically stable if no diffusion, so
 $\text{tr}(\mathbf{J}) < 0$ and $\det(\mathbf{J}) > 0$
2. The uniform steady-state is unstable when diffusion is included, so
 $K > 0$ and $4D_1D_2\det(\mathbf{J}) < K^2$

When a Turing instability occurs:

Activator: autocatalytic and slowly diffusing

Inhibitor: rapidly diffusing