Summary for Nonlinear IVPs

Single Equations

$y' = f(y)$

Steady-state: $f(\bar{y}) = 0$

Stability: The steady-state is asymptotically stable if $f'(\bar{y}) < 0$ and it is unstable if $f'(\bar{y}) > 0$.

Systems of Two Equations

$x' = f(x, y)$
$y' = g(x, y)$

Steady-state: $f(\bar{x}, \bar{y}) = 0$, $g(\bar{x}, \bar{y}) = 0$

Stability: The Jacobian $\mathbf{J}$ is

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\[2ex] \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}.$$

The trace of $\mathbf{J}$ is

$$\mathrm{tr}(\mathbf{J}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

the determinant of $\mathbf{J}$ is

$$\det(\mathbf{J}) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

and the eigenvalues of $\mathbf{J}$ are

$$r_{\pm} = \frac{1}{2}\Big(\mathrm{tr}(\mathbf{J}) \pm \sqrt{[\mathrm{tr}(\mathbf{J})]^2 - 4\det(\mathbf{J})}\Big)$$

In the theorems below, $\mathbf{J}$ is evaluated at $(\bar{x}, \bar{y})$.

Theorem 1: The steady-state is asymptotically stable if $\det(\mathbf{J}) > 0$ and $\mathrm{tr}(\mathbf{J}) < 0$. It is unstable if $\det(\mathbf{J}) < 0$ or $\mathrm{tr}(\mathbf{J}) > 0$.

Suppose the steady-state depends on a parameter λ, and it changes stability at $\lambda = \lambda_b$. Also, it is assumed that $\mathrm{tr}(\mathbf{J})$ and $\det(\mathbf{J})$ are smooth functions of λ.

Theorem 2 (Hopf Bifurcation Theorem): If

1. $[\mathrm{tr}(\mathbf{J})]^2 < 4\det(\mathbf{J})$ at $\lambda = \lambda_b$
2. $\frac{d}{d\lambda}\mathrm{tr}(\mathbf{J}) \neq 0$ at $\lambda = \lambda_b$

then a limit cycle solution appears as λ passes through $\lambda = \lambda_b$. The limit cycle solution is on the side opposite of the side where the steady-state is stable.

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