Steady-States and Stability for Difference Equations

Single Equations

$x_{n+1} = f(x_n)$

Steady-state: $\bar{x} = f(\bar{x})$

Stability: The steady-state is asymptotically stable if $|f'(\bar{x})| < 1$ and it is unstable if $|f'(\bar{x})| > 1$.

Systems of Two Equations

$x_{n+1} = f(x_n, y_n)$
$y_{n+1} = g(x_n, y_n)$

Steady-state: $\bar{x} = f(\bar{x}, \bar{y})$, $\bar{y} = g(\bar{x}, \bar{y})$

Stability: The Jacobian $\mathbf{J}$ is

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}.$$

The trace of $\mathbf{J}$ is

$$\mathrm{tr}(\mathbf{J}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

and the determinant of $\mathbf{J}$ is

$$\det(\mathbf{J}) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

In the theorem below, $\mathbf{J}$ is evaluated at $(\bar{x}, \bar{y})$.

Theorem: The steady-state is asymptotically stable if $|\mathrm{tr}(\mathbf{J})| < 1 + \det(\mathbf{J})$ and $\det(\mathbf{J}) < 1$. It is unstable if $|\mathrm{tr}(\mathbf{J})| > 1 + \det(\mathbf{J})$ or $\det(\mathbf{J}) > 1$.