HW6.1

Solutions to HW6

1.a) u = prey v = predator

=>

$$\partial_t u = D_1 \partial_x^2 u + au - buv$$
$$\partial_t v = \underbrace{D_2 \partial_x^2 v}_{\text{diffusion terms}} \underbrace{- cv + duv}_{\text{Q's come directly from pred-prey eqs}}$$

b) flux for prey $= \underbrace{-D_1 \partial_x v}_{\text{moves in opposite direction of pred. increase}}$

flux for predator $= \underbrace{D_2 \partial_x u}_{\text{moves in same direction as prey increase}}$

=>

$$\partial_t u = D_1 \partial_x^2 v + au - buv$$
$$\partial_t v = -D_2 \partial_x^2 u - cv + duv$$

c) the reaction is $A + 2B \to C + 2D$

$r = -k A B^2$

=>

$$\partial_t A = D_1 \partial_x^2 A - kAB^2$$
$$\partial_t B = D_2 \partial_x^2 B - 2kAB^2$$
$$\partial_t C = D_3 \partial_x^2 C + kAB^2$$
$$\partial_t D = D_4 \partial_x^2 D + 2kAB^2$$

2. $\partial_t b = \partial_x \left(D \partial_x b - \frac{\alpha b}{n} \partial_x n \right) = -\partial_x J_1 + Q_1$

$\partial_t n = -kb = -\partial_x J_2 + Q_2$

$J_2 = 0$: food does not move

$Q_2 = -kb$: food disappears at rate propor. to bacteria (this is typical)

$Q_1 = 0$: bacteria not dieing or being born

HW6.2

$$J_1 = \underbrace{-D\partial_x b}_{\text{standard diffusion}} + \frac{\alpha b}{n}\partial_x n$$

standard diffusion

bacteria move in direction of increasing food ($\alpha > 0$)

if food increases where bacteria is located, its movement is reduced

flux increases as bacteria pop increases

3. a) $c(1-c^2) = 0 \Rightarrow c = 0, \pm 1$

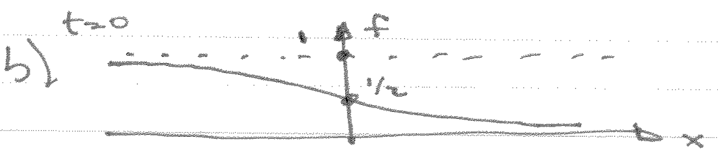
$f(c) = c(1-c^2) \Rightarrow f' = 1 - 3c^2$

$c = 0: \; f' = 1 > 0 \Rightarrow$ unstable

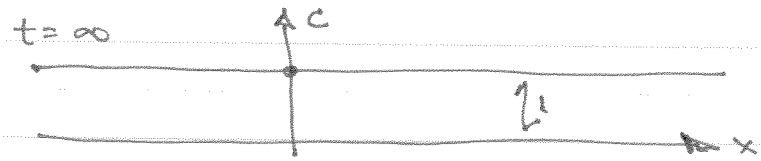
$c = 1: \; f' = -2 \Rightarrow$ stable

$c = -1: \; f' = -2 \Rightarrow$ stable

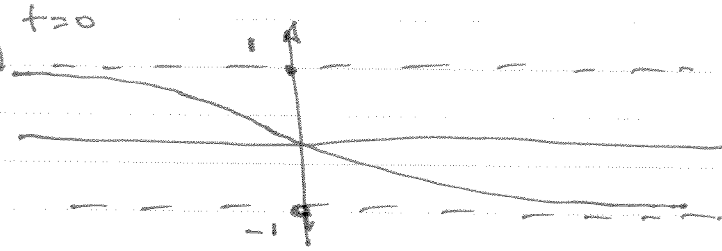
b)



all pts move upwards towards $c = 1$



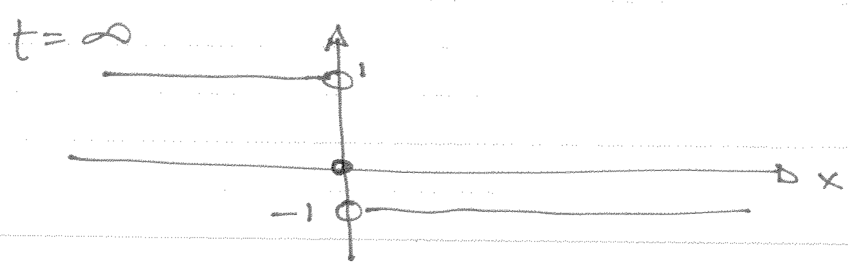
c)



$x < 0$: pts move upwards $c = 1$

$x > 0$: pts move downwards to $c = -1$

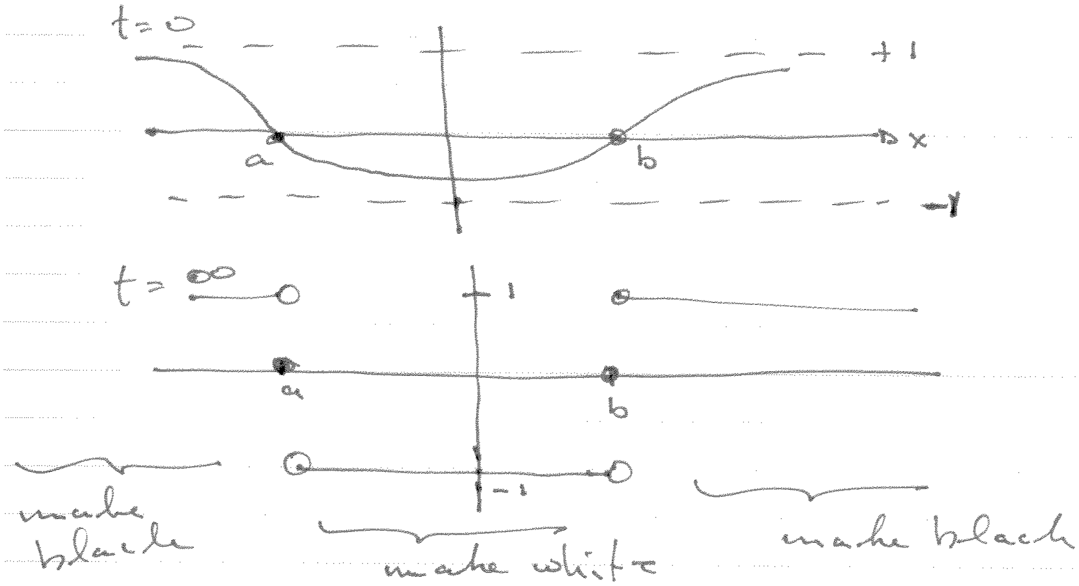
$x = 0$: stays at $c = 0$



~~d) To produce a belted cow, you need a f(x) with three two zeros (2 stable — one unstable)~~

d) You need a f(x) with two zeros — like the following

HW6.3



4. a) The reactions are

$$S + E \rightleftarrows C \rightarrow E + P$$

$\Rightarrow$

$$\frac{\partial S}{\partial t} = D_1 \partial_x^2 S - k_1 S E + k_{-1} C$$

$$\frac{\partial E}{\partial t} = D_2 \partial_x^2 E - k_1 S E + k_{-1} C + k_2 C$$

$$\frac{\partial C}{\partial t} = D_3 \partial_x^2 C + k_1 S E - k_{-1} C - k_2 C$$

$$\frac{\partial P}{\partial t} = D_4 \partial_x^2 P + k_2 C$$

b) $D_1 \partial_x S = 0$ at $x = a, b$

$D_2 \partial_x E = 0$ " "

$D_3 \partial_x C = 0$ " "

$D_4 \partial_x P = 0$ " "

c) Assuming S, E, C & P are constants,

$$-k_1 S E + k_{-1} C = 0$$

$$-k_1 S E + (k_{-1} + k_2) C = 0$$

$$k_1 S E - (k_{-1} + k_2) C = 0$$

$$k_2 C = 0$$

$\Rightarrow$

$$C = 0 \quad \text{and} \quad S \cdot E = 0$$

Now, note by adding original E & C eq

$$\partial_t (E+C) = \partial_x^2 (D_2 E + D_3 C)$$

$\Rightarrow$

$$\frac{d}{dt}\int_a^b (E+C)\,dx = \int_a^b \partial_x^2 (D_2E + D_3 C)\,dx$$
$$= D_2 \partial_x E + D_3 \partial_x C \Big|_{x=a}^{b} = 0$$

$\Rightarrow$

$$\frac{d}{dt}\int_a^b (E+C)\,dx = 0$$

$\Rightarrow$

$$\int_a^b (E+C)\,dx = \int_a^b (E_0 + C_0)\,dx$$
$$= (b-a)(E_0 + C_0)$$

& assume E_0 & C_0 are constant

steady-states: E, C cnst $\Rightarrow$

$$E + C = E_0 + C_0$$

& $C = 0$

$$E = E_0 + C_0$$

~~Now~~, Since $S \cdot E = 0$ then it follows that $S = 0$.

5. a) $\frac{u^2}{v} - bu = 0$ $\Rightarrow$ $u = 0$ & $v = 0$ ← not allowed because eqs not defined there

$u^2 - v = 0$ or $u = bv \Rightarrow v = \frac{1}{b}u \Rightarrow \frac{1}{b}u = u^2$

$\Rightarrow u = \frac{1}{b}$ and $v = \frac{1}{b^2}$

b) $D_1 = D_2 = 0$: $f = \frac{u^2}{v} - bu$ $g = u^2 - v$

$\Rightarrow$

$f_u = \frac{2u}{v} - b$ $f_v = -\frac{u^2}{v^2}$ $g_u = 2u$ $g_v = -1$

$\Rightarrow$

$$J = \begin{pmatrix} \frac{2u}{v} - b & -u^2/v^2 \\ 2u & -1 \end{pmatrix}$$

& $u = 1/b$, $v = \frac{1}{b^2}$

$$= \begin{pmatrix} b & -b^2 \\ 2/b & -1 \end{pmatrix}$$

$\Rightarrow$

$\mathrm{tr}\, J = -1 + b$ $\det J = b$

$\therefore$ for stability we need $-1 + b < 0$ and $b > 0$

$\therefore$ $0 < b < 1$

c) $K = D_1\cdot(-1) + b\cdot D_2 = -D_1 + b\cdot D_2$

$K>0$: $D_1 < b\cdot D_2$

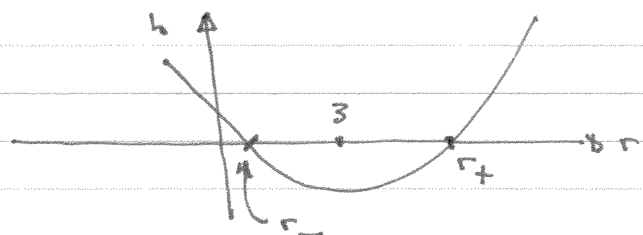
$4D_1D_2\cdot \det J < K^2$: $4D_1D_2\cdot b < (-D_1+bD_2)^2$

d) $4D_1D_2\cdot \det J < K^2 \Rightarrow D_2^2b^2 - 6D_1D_2b + D_1^2 > 0$

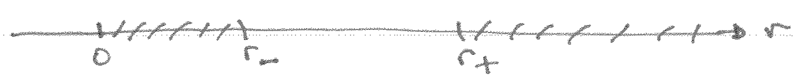
$\Rightarrow \underbrace{r^2 - 6r + 1}_{\text{quadratic fn of } r} > 0$ $\qquad r = \frac{D_2 b}{D_1}$

$h(r) = r^2 - 6r + 1 \qquad h' = 2r - 6 = 0 \Rightarrow r = 3$

$h(0) = 1 \qquad h(3) = 9 - 18 + 1 = -8$



$h = 0 \Rightarrow \quad r_\pm = \frac{1}{2}\left(6 \pm \sqrt{36-4}\right) = 3 \pm 2\sqrt{2}$



} regions where $h>0$

From part (c) we need $D_1 < b\cdot D_2 \Rightarrow r > 1$

this means we need $r > r_+ \Rightarrow 3 + 2\sqrt{2} < \frac{D_2 b}{D_1}$

$$\Rightarrow D_1 < b\cdot D_2 \frac{1}{3+2\sqrt{2}}\,\frac{3-2\sqrt{2}}{3-2\sqrt{2}} = (3-2\sqrt{2})\cdot bD_2$$

e) u is the activator because its eq. contains the required autocatalytic term (u^2/v) — this makes v the inhibitor