Solutions to HW4

1. a) $(\theta, v) = (\bar\theta, \bar v) \Rightarrow \begin{array}{l} \bar v = 0 \\ -\kappa \bar v - \sin\bar\theta = 0 \end{array} \Rightarrow \begin{array}{l} \sin\bar\theta = 0 \\ \theta = 0, \pi \end{array}$

b) $\begin{array}{l} f = v \\ g = -\kappa v - \sin\theta \end{array} \Rightarrow J = \begin{pmatrix} f_\theta & f_v \\ g_\theta & g_v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos\theta & -\kappa \end{pmatrix}$

$\operatorname{tr} J = -\kappa \quad , \quad \det J = \cos\theta$

$(\bar\theta, \bar v) = (0, 0)$: $\operatorname{tr} = -\kappa$, $\det = 1 > 0 \Rightarrow$ asy. stable

$(\bar\theta, \bar v) = (\pi, 0)$: $\operatorname{tr} = -\kappa$, $\det = -1 < 0 \Rightarrow$ unstable

2. $N_1' = rN_1 \cdot \left[1 - \frac{N_1}{K_1 + \alpha N_2}\right] = rN_1 - \frac{rN_1^2}{K_1 + \alpha N_2}$

$N_2' = rN_2 \cdot \left[1 - \frac{N_2}{K_2 + \beta N_1}\right] = rN_2 - \frac{rN_2^2}{K_2 + \beta N_1}$

a) For both eqs, the population of the other species decreases the death rate (these are the negative terms in each eq) and the rate gets smaller as the other pop. increases — hence they are deriving mutual benefit from each other

b) $N_1 = 0 \Rightarrow N_2 = 0$ or $1 - \frac{N_2}{K_2 + \beta N_1} = 0 \Rightarrow N_2 = K_2$

$N_1 = K_1 + \alpha N_2 \Rightarrow N_2 = 0$ or $N_2 = K_2 + \beta(K_1 + \alpha N_2)$

$\Rightarrow N_2 = \frac{K_2 + \beta K_1}{1 - \alpha\beta}$ & $\alpha\beta < 1$

$\therefore$ the steady-states are

$(N_1, N_2) = (0,0), (0, K_2), (K_1, 0)$, and

$\left(\frac{K_1 + \beta K_2}{1 - \alpha\beta}, \frac{K_2 + \beta K_1}{1 - \alpha\beta}\right)$

c) this assumption guarantees that there is a steady-state with both values nonzero.

d) $$J = \begin{pmatrix} r\left(1-\frac{N_1}{K_1+\alpha N_2}\right)-\frac{rN_1}{K_1+\alpha N_2} & r\frac{N_1^2\alpha}{(K_1+\alpha N_2)^2} \\ \frac{rN_2^2\beta}{(K_2+\beta N_1)^2} & r\left(1-\frac{N_2}{K_2+\beta N_1}\right)-\frac{rN_2}{(K_2+\beta N_1)^2} \end{pmatrix}$$

$$= \begin{pmatrix} -r & \alpha r \\ \beta r & -r \end{pmatrix} \qquad \frac{N_1}{K_1+\alpha N_2}=1 \text{ and } \frac{N_2}{K_2+\beta N_1}=1$$

$\Rightarrow$

$$\underset{<0}{tr = -2r} \quad \text{and} \quad det = r^2-\alpha\beta r^2 = (1-\alpha\beta)r^2 > 0$$

$\therefore$ steady-state is asy. stable

3. $x' = x[x(1-x)-y]$ $\quad k>0$ and $\mu>0$
$y' = k(x-\frac{1}{\mu})y$

a) First note that the x' equation has multiple interpretations (depending how you write it). One way is

$$x' = \underbrace{x^2(1-x)} - xy$$

for $x<1$ this is the birth rate but for $x>1$ it contributes to the death rate

death rate (usual pred/prey term)

$$y' = \underbrace{-\frac{k}{\mu}y} + \underbrace{kxy}$$

usual death rate for predator

usual birth rate for predator

b) $(x,y)=(\bar{x},\bar{y}) \Rightarrow \bar{x}[\bar{x}(1-\bar{x})-\bar{y}]=0$
$k(\bar{x}-1/\mu)\bar{y}=0$

$\bar{y}=0 \Rightarrow \bar{x}^2(1-\bar{x})=0 \Rightarrow \bar{x}=0,1$

$\bar{x}=\frac{1}{\mu} \Rightarrow \bar{y}=\bar{x}(1-\bar{x})=\frac{1}{\mu}\left(1-\frac{1}{\mu}\right)$

$\therefore$ steady-states are

$(\bar{x},\bar{y}) = (0,0),\ (1,0),\ \left(\frac{1}{\mu}, \frac{1-1/\mu}{\mu}\right)$

c) $f = x^2(1-x) - xy$
$g = k\left(x - \frac{1}{\mu}\right)y$

$$J = \begin{pmatrix} 2x(1-x) - x^2 - y & -x \\ ky & k\left(x - \frac{1}{\mu}\right) \end{pmatrix}$$

$(\bar{x},\bar{y}) = (1,0)$: $\quad J = \begin{pmatrix} -1 & -1 \\ 0 & k(1-1/\mu) \end{pmatrix}$

$tr = -1 + k\left(1 - \frac{1}{\mu}\right) \qquad det = -k\left(1-\frac{1}{\mu}\right)$

so, $\mu > 1 \Rightarrow det < 0$ and $\mu < 1 \Rightarrow det > 0$ and $tr < 0$

$\therefore$ ass stable if $\mu < 1$ and unstable if $\mu > 1$

$(\bar{x},\bar{y}) = \left(\frac{1}{\mu}, \frac{1-1/\mu}{\mu}\right) \Rightarrow \quad J = \begin{pmatrix} \frac{1}{\mu}(1-2/\mu) & -1/\mu \\ \frac{k}{\mu}(1-1/\mu) & 0 \end{pmatrix}$

$tr = \frac{1}{\mu}\left(1 - \frac{2}{\mu}\right)$ and $det = \frac{k}{\mu^2}\left(1 - \frac{1}{\mu}\right)$

so, $\mu < 1 \Rightarrow det < 0 \Rightarrow$ unstable

$1 < \mu < 2 \Rightarrow det > 0$ and $tr < 0 \Rightarrow$ ass. stable

$\mu > 2 \Rightarrow tr > 0 \Rightarrow$ unstable

d) $(\bar{x},\bar{y}) = (1,0)$: $\quad tr^2 - 4det = \left[-1 + k\left(1 - \frac{1}{\mu}\right)\right]^2 - 4\cdot(-k)\left(1 - \frac{1}{\mu}\right)$

$= 1 > 0 \quad \mu = 1$

$\therefore$ no Hopf bif.

$(\bar{x},\bar{y}) = \left(\frac{1}{\mu}, \frac{1-1/\mu}{\mu}\right)$: $\quad tr^2 - 4det = \frac{1}{\mu^2}\left(1 - \frac{2}{\mu}\right)^2 - \frac{4k}{\mu^2}\left(1 - \frac{1}{\mu}\right)$

$\mu = 1 \Rightarrow$ no Hopf bif (see above)
$\mu = 2 \Rightarrow tr^2 - 4det = -4\cdot\frac{k}{4}\cdot\frac{1}{2} < 0$

and
$$\frac{d}{d\mu} f_1 = \frac{-1}{\mu^2}\left(1-\frac{2}{\mu}\right) + \frac{1}{\mu}\cdot\frac{2}{\mu^2} \Big|_{\mu=2}$$
$$= \frac{1}{4} > 0$$
$\therefore$ there is a Hopf bif at $\mu = 2$

Extra Credit

a)
$$S' = \Pi - \overset{①}{\beta S Z} - \overset{②}{\delta S}$$
$$Z' = \overset{①}{\beta S Z} + \overset{④}{\zeta R} - \overset{③}{\alpha S Z}$$
$$R' = \overset{②}{\delta S} + \overset{③}{\alpha S Z} - \overset{④}{\zeta R}$$

① $S + Z \xrightarrow{\beta} ZZ$ } alive + zombie → 2 zombies

② $S \xrightarrow{\delta} R$ } alive becomes dead

③ $S + Z \xrightarrow{\alpha} S + R$ } alive + zombie → alive + dead

④ $R \xrightarrow{\zeta} Z$ } dead becomes zombie

b) There is no "really dead" state – which means a state the zombies can't come back from

One way to fix this is to replace ③ with
$$S + Z \rightarrow S + D$$

where $D \equiv$ really dead