Math in Medicine and Biology F'15
Exam 2 Name: ______________________________

1. Find the solution of

$$y' = \frac{3t^2}{1+2y}$$

that satisfies $y(0) = 0$.

Using separation of variables

$$(1+2y)\,dy = 3t^2\,dt$$

$$\Rightarrow \quad y + y^2 = t^3 + c \qquad \& \quad y(0)=0 \Rightarrow c = 0$$

$$= t^3$$

$$\Rightarrow \quad y^2 + y - t^3 = 0$$

$$\Rightarrow \quad y = \tfrac{1}{2}\left(-1 \pm \sqrt{1+4t^3}\right) \qquad \& \quad y(0)=0 \Rightarrow \text{use + sign}$$

$$= \tfrac{1}{2}\left(-1 + \sqrt{1+4t^3}\right)$$

2. Consider the reactions

$$A \rightleftharpoons C + D, \quad \begin{matrix} k_1 \\ k_{-1} \end{matrix}$$
$$A + B \to 2A + C. \quad k_2$$

a) Write down the dynamical system for these reactions.

$$A' = -k_1 A + k_{-1} CD - k_2 AB + 2k_2 AB$$
$$= -k_1 A + k_{-1} CD + k_2 AB$$
$$B' = -k_2 AB$$
$$C' = k_1 A - k_{-1} CD + k_2 AB$$
$$D' = k_1 A - k_{-1} CD$$

b) Find a conservation law from your equations (you only need to find one). Make sure to explain, or show, how you determine this.

$$A' + C' + 2B' = 0 \implies A + 2B + C = A_0 + 2B_0 + C_0$$

or

$$A' + D' + B' = 0 \implies A + B + D = A_0 + B_0 + D_0$$

3. The Nicholson-Bailey model for a host and parasite is

$$x' = x(3e^{-y} - 1),$$
$$y' = -y + x(1 - e^{-y}).$$

a) Determine which variable is used for the host and which is used for the parasite. Make sure you explain why.

$x = 0 \Rightarrow y' = -y \Rightarrow y$ dies off

conclusion: without x, y dies off

$y = 0$: $x' = 2x \Rightarrow x$ grows

conclusion: x does not need y to survive

conclusion: x is host
y is parasite

b) Determine the steady-state(s).

$x' = 0 \Rightarrow x = 0$ or $3e^{-y} - 1 = 0 \Rightarrow y = \ln 3$

$x = 0$: $-y + x\cdot(1 - e^{-y}) = -y = 0 \Rightarrow y = 0$

$y = \ln 3$: $-y + x(1 - e^{-y}) = -\ln 3 + x \cdot \frac{2}{3} = 0 \Rightarrow x = \frac{3}{2}\ln 3$

so $(0, 0)$ and $(\frac{3}{2}\ln 3, \ln 3)$

c) One of the steady-states is nonzero. Determine if it is stable or unstable. Hint: $\ln 3 \approx 1.1$.

$$J = \begin{pmatrix} 3e^{-y} - 1 & x \cdot (-3e^{-y}) \\ 1 - e^{-y} & -1 + x e^{-y} \end{pmatrix}$$

$$x = \tfrac{3}{2}\ln 3 \qquad y = \ln 3$$

$$= \begin{pmatrix} 0 & -\tfrac{3}{2}\ln 3 \\ \tfrac{2}{3} & -1 + \tfrac{1}{2}\ln 3 \end{pmatrix}$$

$\Rightarrow$

$$\mathrm{tr} = -1 + \underbrace{\tfrac{1}{2}\ln 3}_{\approx\, 0.55} < 0 \qquad \det = \ln 3 > 0$$

$\therefore$ asy. stable

4. A model for action potentials is

$$v' = v(1-v)(v-2) - w + I,$$
$$w' = 2v - w.$$

Note that v and w can be positive or negative (i.e., they are not restricted to being just positive).
a) The nullclines are shown in the figure on the next page (for $I = 2$). Explain how these curves are determined.

set $v' = 0$ and find w,v curve

and set $w' = 0$ and find w,v curve

b) According to the graph, how many steady-states are there, and where are they (or, it) located? Make sure to provide a reason for your answer.

there is one, which is where $w' = 0$ and $v' = 0$, and so its where the nullclines intersect

c) In the figure, for each of the four regions determined by the nullclines, draw an arrow indicating the direction of motion.

d) Suppose v reacts very quickly (in the manner associated with the QSSA). In the figure, assuming that the starting point is $v(0) = 0.5$ and $w(0) = 2.5$, sketch the solution curve.

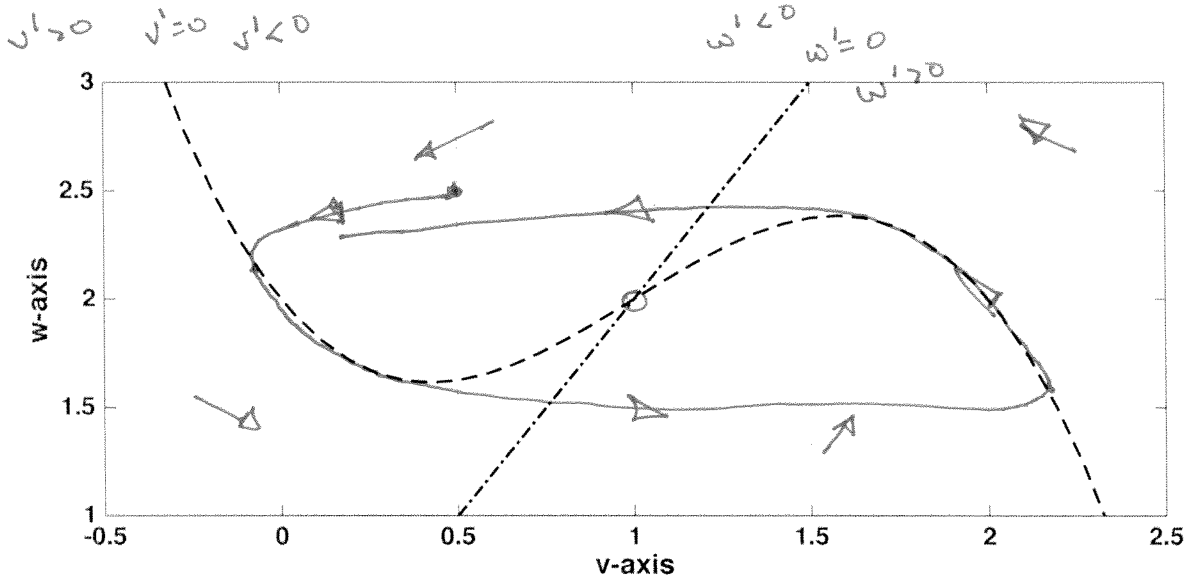

Figure 1: Graph for Problem 4.