Math in Medicine and Biology F'13
Exam 2 Name: ______________________

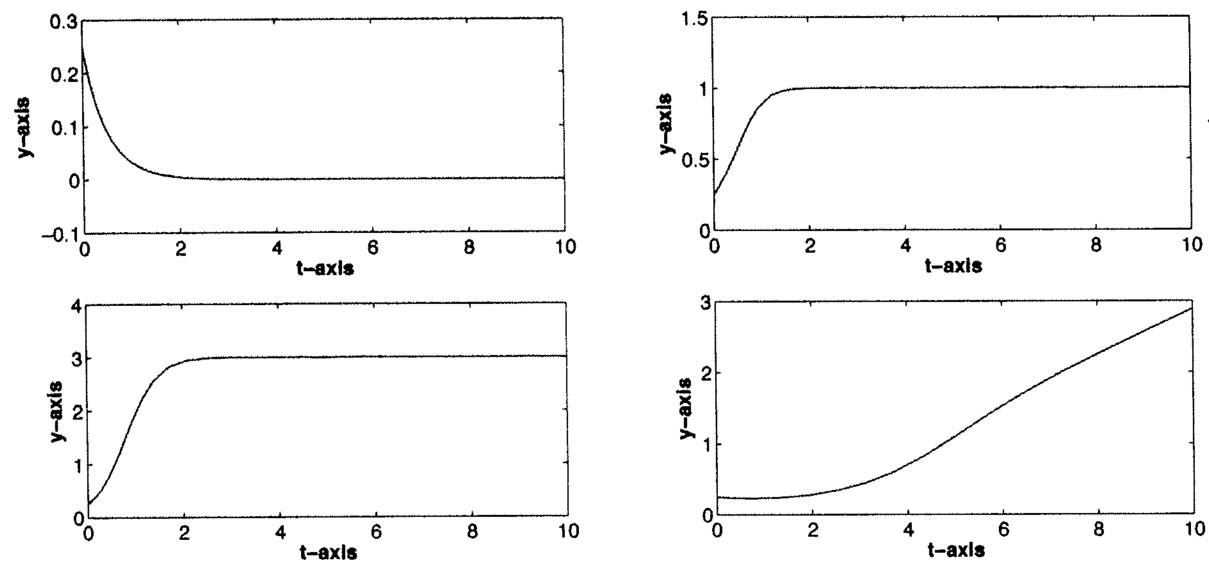
1. Find the solution of $y' = -ty^2$ that satisfies $y(0) = 1$.

$$\frac{dy}{y^2} = -t\,dt \Rightarrow -\frac{1}{y} = -\frac{1}{2}t^2 + C \Rightarrow y = \frac{1}{\frac{1}{2}t^2 - C}$$

$$y(0) = 1 \Rightarrow 1 = \frac{1}{-C} \Rightarrow C = -1$$

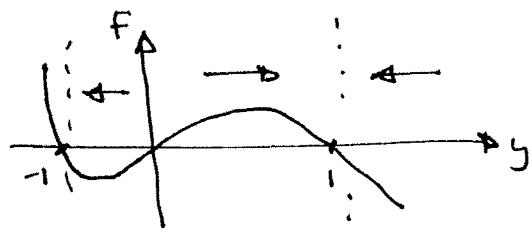
$$\therefore \quad y = \frac{1}{\frac{1}{2}t^2 + 1} = \frac{2}{t^2 + 2}$$

2. One of the graphs below shows the solution of $y' = 2y(1 - y^2)$, with $y(0) = \frac{1}{4}$. Which one is it? You must provide a clear reason for your conclusion, and to answer this you do not need to solve the problem.



this one

Figure 1: Curves for Exercise 2.



$f(y) = 2y(1-y^2)$

$\Rightarrow$ $y = 1$ is an asy. stable steady state (and $y = 0$ is unstable and $y = 3$ is not even a steady-state)

$\therefore$ top right is the choice

3. This problem concerns the two reactions shown below.
a) There are four species. Label them and write down the corresponding reactions.
b) Derive the dynamical system for these reactions.

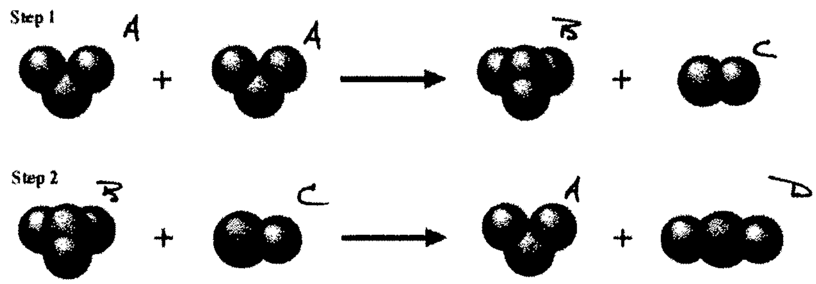


Figure 2: Reactions for Exercise 3.

$$2A \to B + C \qquad k_1 \qquad r_1 = k_1 A^2$$
$$B + C \to A + D \qquad k_2 \qquad r_2 = k_2 BC$$

$\Rightarrow$

$$A' = -2r_1 + r_2 = -2k_1 A^2 + k_2 BC$$
$$B' = r_1 - r_2 = k_1 A^2 - k_2 BC$$
$$C' = r_1 - r_2 = k_1 A^2 - k_2 BC$$
$$D' = k_2 BC$$

4. A model for two species of herbivores (plant eaters) is

$$x' = x(-a + by)$$
$$y' = \frac{y}{1 + cx} - dy$$

where a, b, c, and d are positive constants. Explain the physical significance of each term on the right hand side of these equations (things to think about: birth rate, death rate, cooperation, competition, mutualism, etc).

$x' = -ax + bxy$

ax: this is the (usual) death rate - namely, rate propor. to current pop

bxy: this is the birth rate - propor. (as usual) to current pop. x but also propor. to y — so y's increase birth rate of x's (in sense that larger y => larger birth rate for x) - this is what is expected for mutualism (although the x's do not have the same affect on the y's)

$y' = \frac{y}{1+cx} - dy$

dy: death rate (the usual)

$\frac{y}{1+cx}$: birth rate - note that it decreases as x increases which is typical for competition (although the y's do not have a similar effect on x's)

5. Consider the equations

$$x' = x(-a+y) \quad = \quad -ax + xy$$
$$y' = \frac{y}{1+x} - y^2$$

where a is a positive constant (you can assume $a \neq 1$).
a) Find the steady-states.
b) Determine if each steady-state is asymptotically stable or unstable.

a) $x = 0 \Rightarrow y - y^2 = 0 \Rightarrow y = 0$ or 1

$a = y \Rightarrow \frac{1}{1+x} = a \Rightarrow 1 + x = \frac{1}{a} \Rightarrow x = \frac{1}{a} - 1$

$\therefore$ $(0,0)$, $(0,1)$ and $(\frac{1}{a} - 1, a)$

b) $$J = \begin{pmatrix} -a+y & x \\ \frac{-y}{(1+x)^2} & \frac{1}{1+x} - 2y \end{pmatrix}$$

$(0,0)$: $J = \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ $tr = 1 - a$
$det = -a < 0 \leftarrow \therefore$ unstable

$(0,1)$: $J = \begin{pmatrix} -a+1 & 0 \\ -1 & -1 \end{pmatrix} \Rightarrow$ $tr = -a < 0$
$det = a - 1 > 0$ if $a > 1$
So ass. stable if $a > 1$
and unstable if $0 < a < 1$

$(\frac{1}{a} - 1, a)$: $J = \begin{pmatrix} 0 & \frac{1}{a} - 1 \\ -a^3 & -a \end{pmatrix} \Rightarrow$ $tr = -a < 0$
$det = -a^3(1 - \frac{1}{a}) > 0 \iff$
$1 < \frac{1}{a} \iff a < 1$
$\therefore$ ass stable if $0 < a < 1$
and unstable if $a > 1$