Math in Medicine and Biology F'15
Exam 1 Name: ______________________

1. For the following two problems, find the solution.

a) $x_n = 3x_{n-1}$, with $x_0 = 3$

$x_n = r^n \Rightarrow r = 3 \Rightarrow x_n = x_0 \cdot 3^n$

$= 3 \cdot 3^n = 3^{n+1}$

b) $x_{n+1} = -y_n$
$y_{n+1} = x_n + 4y_n$, where $x_0 = 1$ and $y_0 = 0$.

$x_n = -y_{n-1} \Rightarrow y_{n+1} = -y_{n-1} + 4y_n$

$y_n = r^n \Rightarrow r^2 = -1 + 4r$

$r^2 - 4r + 1 = 0$

$\Rightarrow r_\pm = \frac{1}{2}\left(4 \pm \sqrt{16-4}\right)$

$= 2 \pm \sqrt{3}$

general sol:

$y_n = \alpha r_+^n + \beta r_-^n$

$x_n = -y_{n-1} = -\alpha r_+^{n-1} - \beta r_-^{n-1}$

$y_0 = 0 \Rightarrow \alpha + \beta = 0 \Rightarrow \beta = -\alpha$

$x_0 = 1 \Rightarrow y_1 = x_0 + 4y_0 = 1 \Rightarrow \alpha r_+ + \beta r_- = 1$

$\alpha = \frac{1}{r_+ - r_-} = \frac{1}{2\sqrt{3}}$

$\beta = -\frac{1}{2\sqrt{3}}$

2. For the following two problems, find the steady-state(s) and determine their stability.

a) $x_{n+1} = \dfrac{2x_n}{1+4x_n}$

$$x_n = \bar{x} \Rightarrow \bar{x} = \frac{2\bar{x}}{1+4\bar{x}}$$

$$\bar{x} = 0 \quad \underline{\text{or}} \quad 1 = \frac{2}{1+4\bar{x}} \Rightarrow 1 + 4\bar{x} = 2$$

$$\bar{x} = \frac{1}{4}$$

$$f(x) = \frac{2x}{1+4x} \Rightarrow f' = \frac{2(1+4x) - 2x \cdot 4}{(1+4x)^2} = \frac{2}{(1+\cancel{4}x)^2}$$

$\bar{x} = 0$: $f'(0) = 2 > 1 \Rightarrow$ unstable

$\bar{x} = \frac{1}{4}$: $f'(\frac{1}{4}) = \frac{2}{2^2} = \frac{1}{2} < 1 \Rightarrow$ stable

b) $x_{n+1} = 2x_n - y_n$
$y_{n+1} = x_n(9 - y_n^3)$

$x_n = \bar{x}$ & $y_n = \bar{y}$ $\Rightarrow$ $\bar{x} = 2\bar{x} - \bar{y}$
$\bar{y} = \bar{x}(9 - \bar{y}^3)$

$\Rightarrow$ $\bar{x} = \bar{y}$
$\bar{y} = \bar{y}(9 - \bar{y}^3)$

$\Rightarrow$ $\bar{y} = 0$ or $1 = 9 - \bar{y}^3$
$\bar{y}^3 = 8 \Rightarrow \bar{y} = 2$

steady-states: $(0,0)$ & $(2,2)$

$f(x,y) = 2x - y$
$g(x,y) = x(9 - y^3)$ $\Rightarrow$ $f_x = 2$ $\quad f_y = -1$
$g_x = 9 - y^3$ $\quad g_y = -3xy^2$

$\bar{x} = \bar{y} = 0 \Rightarrow J = \begin{pmatrix} 2 & -1 \\ 9 & 0 \end{pmatrix} \Rightarrow$ $tr = 2$
$det = 9$ & violates $det < 1$ condition
$\therefore$ unstable

$\bar{x} = \bar{y} = 2 \Rightarrow J = \begin{pmatrix} 2 & -1 \\ 1 & -24 \end{pmatrix}$

$\left.\begin{array}{l} tr = -22 \\ det = -48 + 1 = -47 \end{array}\right]$ violates $|tr| < 1 + det$ condition $\Rightarrow$ unstable

3. Let x_n denote the population of hummingbirds and y_n the population of flowers at time t_n. Also, suppose that

$$x_{n+1} = \alpha x_n$$
$$y_{n+1} = \beta y_n$$

Note that α and β are possibly functions of x_n and y_n (so, it is not possible to solve this problem by hand). The following are observed:
i) When there is a large number of hummingbirds, their population decreases.
ii) When there is a large number of flowers, their population decreases.
iii) The hummingbirds and flowers are mutually beneficial, which means that the increase in the population for one of them results in the increase in the population of the other.
Which of the following comes closest to satisfying these observations? Make sure to explain why.

a) $\alpha = y_n$ $\quad\beta = x_n$

b) $\alpha = \dfrac{y_n}{1+x_n}$ $\quad\beta = \dfrac{x_n}{1+y_n}$

c) $\alpha = x_n^2 y_n$ $\quad\beta = x_n y_n^2$

d) $\alpha = x_n + y_n$ $\quad\beta = x_n + y_n$

Note i) $\Rightarrow$ $\alpha < 1$ for large x_n
ii) $\Rightarrow$ $\beta < 1$ for large y_n
iii) $\Rightarrow$ $\alpha \uparrow$ if $y_n \uparrow$ and $\beta \uparrow$ if $x_n \uparrow$

a), c) & d) violate i & ii)
b) satisfies all three $\leftarrow$ best choice