Math in Medicine and Biology F'13

Exam 1 Name: ______________________

1. For the following two problems, find the solution.

a) $x_n = -2x_{n-1}$ with $x_0 = 1$.

$X_n = r^n \Rightarrow r^n = -2r^{n-1} \Rightarrow r = -2$

$\Rightarrow$

$X_n = X_0 \cdot (-2)^n$ & $X_0 = 1$

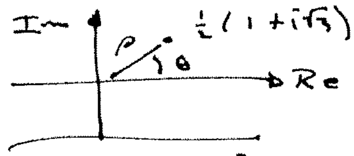
$= (-2)^n$

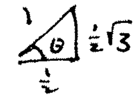
b) $x_{n+1} = x_n - x_{n-1}$ with $x_0 = 1$ and ~~$x_0 = 0$.~~ $x_1 = 0$

$$x_n = r^n \Rightarrow r^{n+1} = r^n - r^{n-1}$$

$$\Rightarrow r^2 - r + 1 = 0$$

$$\Rightarrow \underline{r_{\pm}} = \tfrac{1}{2}\left(1 \pm \sqrt{1-4}\right) = \tfrac{1}{2}\left(1 \pm i\sqrt{3}\right)$$





$\sin\theta = \frac{1}{2}\sqrt{3}$

$$\rho = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\sqrt{3}\right)^2} = 1$$

$$\theta = \frac{\pi}{3} \qquad r_{\pm} = e^{\pm i \frac{\pi}{3}}$$

$\Rightarrow$

$$x_n = A r_+^n + B r_-^n$$

$$= \alpha \cdot \cos\left(\frac{n\pi}{3}\right) + \beta \cdot \sin\left(\frac{n\pi}{3}\right)$$

$x_0 = 1 \Rightarrow \quad 1 = \alpha$

$x_1 = 0 \Rightarrow \quad 0 = \alpha \cdot \cos\frac{\pi}{3} + \beta \cdot \sin\frac{\pi}{3} = \alpha \cdot \frac{1}{2} + \beta \cdot \frac{1}{2}\sqrt{3}$

$\Rightarrow$

$$\beta = -\frac{1}{\sqrt{3}}$$

$$\therefore \quad x_n = \cos\left(\frac{n\pi}{3}\right) - \frac{1}{\sqrt{3}} \sin\left(\frac{n\pi}{3}\right)$$

2. For the following two problems, find the steady-states(s) and determine their satbility.
a) $x_{n+1} = -3x_n + x_n^2$

steady-state(s): $x_n = \bar{x} \Rightarrow \bar{x} = -3\bar{x} + \bar{x}^2$

$\Rightarrow \bar{x} = 0$ $\underline{\text{or}}$ $1 = -3 + \bar{x} \Rightarrow \bar{x} = 4$

stability: $f(x) = -3x + x^2 \Rightarrow f' = -3 + 2x$

$\bar{x} = 0$: $f'(\bar{x}) = -3 \Rightarrow |f'(\bar{x})| > 1 \Rightarrow$ unstable

$\bar{x} = 4$: $f'(\bar{x}) = -3 + 8 = 5 \Rightarrow |f'(\bar{x})| > 1 \Rightarrow$ unstable

b) $x_{n+1} = -y_n$

$$y_{n+1} = \frac{2y_n}{1+x_n}$$

steady-state(s): $x_n = \bar{x}$ & $y_n = \bar{y}$ $\Rightarrow$

$$\bar{x} = -\bar{y}$$
$$\bar{y} = \frac{2\bar{y}}{1+\bar{x}}$$

$\bar{y} = 0 \Rightarrow \bar{x} = 0$ $\quad$ <u>or</u> $\quad$ $\bar{y} \neq 0 \Rightarrow 1 = \frac{2}{1+\bar{x}} \Rightarrow 1 + \bar{x} = 2$

$\Rightarrow \bar{x} = 1$ and $\bar{y} = -\bar{x} = -1$

stability:

$f(x,y) = -y \Rightarrow f_x = 0 \qquad f_y = -1$

$g(x,y) = \frac{2y}{1+x} \Rightarrow g_x = \frac{-2y}{(1+x)^2} \qquad g_y = \frac{2}{1+x}$

$\bar{x} = \bar{y} = 0$: $\quad J = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix} \Rightarrow$ evalues are $0, 2$

↑ bigger than one

$\therefore$ unstable

$\bar{x} = 1, \bar{y} = -1$: $\quad J = \begin{pmatrix} 0 & -1 \\ \frac{2}{2^2} & \frac{2}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{1}{2} & 1 \end{pmatrix}$

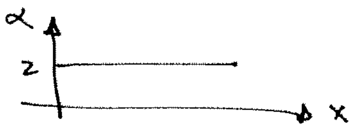
$tr = 1 \qquad det = \frac{1}{2}$

since $det < 1$ <u>and</u> $tr < 1 + det \Rightarrow$ asy. stable

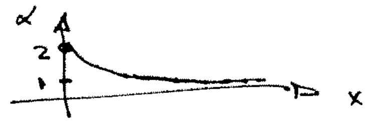
3. a) The equation for a population is $x_{n+1} = \alpha x_n$, where α is positive. It is found that the population decreases if the population is large but increases when the population is small. Which α below can account for this observation? Make sure to state why (or how).

i) $\alpha = 2$ ii) $\alpha = 2\exp(-x_n)$ iii) $\alpha = \dfrac{2x_n}{1 + x_n}$ iv) $\alpha = 2x_n$

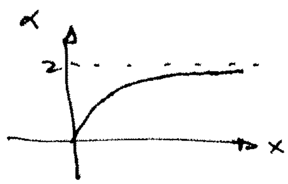
i)



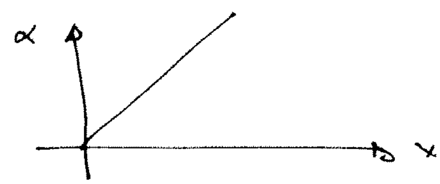
ii)



iii)



iv)



observations:

1) pop decreases $\Rightarrow$ $\alpha < 1$ for large x

2) pop increases $\Rightarrow$ $\alpha > 1$ for small x

the only fn that does this is (ii)

b) Let x_n be the population of deer and y_n the population of bushes. The following is observed:
i) Over a time-step each deer eats 5 bushes and produces 2 babies.
ii) Over a time-step a bush produces 3 baby bushes. You can assume this happens before the deer start eating them.
A model for this is below. Based on the stated observations, what are the values of the constants a, b c, and d? Note it is possible that one or more of the values is indeterminate from the given observations. Also, make sure to provide reasons for your answers.

$$x_{n+1} = ax_n + by_n$$
$$y_{n+1} = cx_n + dy_n$$

observations:
i) each deer eats 5 bushes
$\Rightarrow c = -5$ (this assumes babies don't eat bushes - if they do, then $c = -5 \cdot 3$)

ii) each deer produces 2 babies $\Rightarrow a = 1 + 2 = 3$

iii) each bush produces 3 babies $\Rightarrow d = 1 + 3 = 4$

iv) no info is provided to find b