

Appendix F
Answers to Selected Exercises
Introduction to Perturbation Methods
by M. H. Holmes
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Chapter 1
Introduction to Asymptotic Approximations

Order Symbols

1.1

a) ii) $\alpha \leq 2$, v) $\alpha < 1$

1.2

c) $f = -g = 1/\varepsilon$ and $\varepsilon_0 = 0$

1.6

d) $\phi_1 \ll \phi_5 \ll \phi_2 \ll \phi_3 \ll \phi_4 \ll \phi_6$

e) $\phi_4 \ll \phi_1 \ll \phi_2 \ll \phi_3$

g) not possible

1.7

b) $2^{3/2}(1 + 3\varepsilon^2/8)$

c) $\sinh(1) + \varepsilon x \cosh(1)/2 - \varepsilon^2 x^2/(8e)$

d) $e^x(1 - \varepsilon x^2/2)$

g) $1 + \varepsilon n(n + 1)/2$

1.8

b) $f \sim \frac{\pi}{2\varepsilon} - 1$

c) $f \sim -g(0) \ln(\varepsilon) + \int_0^1 \frac{g(x) - g(0)}{x} dx$

1.10

b) $f = 1/\varepsilon$ and $\phi = -1/\varepsilon$; must have $f - \phi = o(1)$; yes, because given any $\delta > 0$ we can get $|f - \phi| < \delta|\phi|$. But, $\phi = O(1) \Rightarrow |\phi| \leq K$ and so given any

$\delta > 0$, we get that $|f - \phi| < \delta$.

1.11

b) $f = 1 + \varepsilon^2$, $g = -1 + \varepsilon^2$, $\phi = 1$ and $\varphi = -1$ with $\varepsilon_0 = 0$

1.15

b) $\llbracket S \rrbracket \sim c_v \gamma (\gamma^2 - 1) \varepsilon^3 / 12 + O(\varepsilon^4)$

1.18

c) $-2 + \varepsilon$, $1 + \varepsilon^2$

e) $1/3 + \varepsilon/81$, $\pm \sqrt{3/\varepsilon} - 1/6$

f) $\varepsilon + \varepsilon^5$, $(\pm 1 - \varepsilon^2/2)/\varepsilon$

g) $\pm \sqrt{e^{1/2} - 1} - \varepsilon [2 \pm (1 - e^{-1/2})^{-1/2}] / 8$

h) $1 - \sqrt{3}\varepsilon/2$, $-1 + \varepsilon/2$

i) setting $f = x - \int_0^\pi \exp(\varepsilon \sin(x+s)) ds$ then $f(0) < 0 < f(2\pi)$ for $f' > 0$ for $0 < \varepsilon < 1/3$ means only one solution; $x \sim \pi - 2\varepsilon$

m) $x \sim \pm \sqrt{-\ln(\varepsilon)} (1 + \varepsilon/(2 \ln^2 \varepsilon))$

n) $x \sim \varepsilon(1 + \alpha\varepsilon + \alpha(\alpha - 1)\varepsilon^2 + \dots)$

o) $x \sim \varepsilon/p_0^{1/3} - p_0'\varepsilon^2/(5p_0^{5/3})$

p) $x_l \sim \varepsilon(1 + \varepsilon + \dots)$ and $x_r \sim z_0 + \ln(z_0)$, where $z_0 = -\ln(\varepsilon)$

1.22

a) $x \sim k + \varepsilon x_1$ where $x_1 = (-1)^k 210k^{19} / [(k-1)!(20-k)!]$

b) $\varepsilon < 2.2 \times 10^{-11}$

1.23

a) $x \sim -\cos(1) + 2\varepsilon \sin(1) \cos(1) + \dots$

b) $P \sim 2\pi(1 + 100\varepsilon^2)$

1.25

b) $E \sim M + \varepsilon \sin(M) + \frac{1}{2}\varepsilon^2 \sin(2M) + \frac{1}{8}\varepsilon^3 (3 \sin(3M) - \sin(M))$

1.27

b) $k(s) \sim \varepsilon/\beta$

1.28

b) $x_s \sim \varepsilon \ln(3/y_0) - \varepsilon^2 y_1/y_0$ where $y_0 = y(0) \approx 1.3$ and $y_1 = \frac{1}{2} \ln(3/y_0) \ln(y_0/(2-y_0))$

1.29

a) $\lambda \sim \lambda_0 + \varepsilon \lambda_1$ where λ_0 is an eigenvalue for \mathbf{A} and $\lambda_1 = \mathbf{x}_0^T \mathbf{D} \mathbf{x}_0 / \mathbf{x}_0 \cdot \mathbf{x}_0$

b) $\lambda \sim \lambda_0 + \varepsilon\lambda_1$ where $\mathbf{D}\mathbf{x} = \lambda_1\mathbf{x}$

1.30

a) $\mathbf{A}^{-1} - \varepsilon\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$

1.31

a) $\mathbf{A}^\dagger + \varepsilon(\mathbf{K}^{-1}\mathbf{B}^T\mathbf{P}_\perp - \mathbf{A}^\dagger\mathbf{B}\mathbf{A}^\dagger)$ where $\mathbf{K} = \mathbf{A}^T\mathbf{A}$ and $\mathbf{P}_\perp = \mathbf{I} - \mathbf{A}\mathbf{A}^\dagger$

1.34

a) $\tau_h \sim 2 + 4\varepsilon$

1.35

b) $y \sim y_0 + \varepsilon y_1$ where $y_0 = -\tau/\alpha + (1 + \alpha)(1 - e^{-\alpha\tau})/\alpha^2$ and

$$\alpha y_1 = \int_0^\tau f(s)ds - e^{-\alpha\tau} \int_0^\tau f(r)e^{\alpha r} dr$$

c) Method 1: if $y_0(\tau_0) = 0$ with $\tau_0 > 0$ and $\alpha > 0$, then i) $y_0(2) > 0$, ii) $\frac{d}{d\alpha}\tau_0 < 0$ if $\tau_0 > 1$, and iii) $\alpha = 1 \Rightarrow 1.5 < \tau_0 < 1.6$. Thus, $1 < \tau_0 < 2 \forall \alpha > 0$;
Method 2: $\alpha \ll 1 \Rightarrow \tau \sim 2 - 2\alpha/3 \Rightarrow$ decrease

1.36

$\lambda \sim n\pi(1 - \kappa_n\varepsilon)$ where $\kappa_n = \int_0^1 \mu(x) \sin^2(n\pi x) dx$

1.37

a) $y \sim A \cos(n\pi x)$ where $\omega = n\pi + \varepsilon^2(2 - \alpha^2 - \alpha^4)n\pi A^2/(16\alpha^2)$ for $n = 1, 2, 3, \dots$

1.40

a) $\lambda \sim a\lambda_0 + a^2\lambda_1$ where $\lambda_0 = K(0, 0)$ for $\lambda_1 = \frac{1}{2}[K_x(0, 0) + K_s(0, 0)]$, and $y \sim y_0(x) + ay_1(x)$ where $y_0(x) = K(x, 0)/K(0, 0)$, and $y_1 = [K_s(x, 0) - 2\lambda_1 y_0(x)]/(2\lambda_0)$

c) λ as in (a) and $\phi \sim 1 + aK_x(0, 0)(\xi - 1/2)/K(0, 0)$

1.41

c) $E_1 = \int_{-\infty}^{\infty} V_1 \psi_0^2 dx$ and $E_2 = -\int_{-\infty}^{\infty} \psi_0^{-2} \left(\int_{-\infty}^x (V_1 - E_1) \psi_0^2 ds \right)^2 dx$

d) $\psi_0 = A_0 \exp(-\frac{1}{2}E_0 x^2)$, $A_0 = (\lambda/\pi)^{1/4}$

??

$\partial_t c_0 + f(c_0) = \alpha g(t)$ where $c_0(0) = 0$ and $\alpha = |\partial\Omega_1|/|\Omega|$

??

a) ii) $L\partial_{n^*} = \partial_z + \varepsilon(f\partial_z^2 - f_x\partial_x - f_y\partial_y) + O(\varepsilon^2)$ evaluated at $z = 0$

1.45

b) $\rho \sim \varepsilon^{2n}[(\beta - 1)(1 - s^2)/2]^n$ where $r = 1 + \varepsilon s$

Chapter 2

Matched Asymptotic Expansions

Introductory Example

2.1

a) $y \sim ax + (1 - a)(1 - e^{-x/\varepsilon})$

2.2

a) BL at $x = 0$, $y \sim 1/\sqrt{3+x} - e^{-2\bar{x}/\sqrt{3}}$ where $\bar{x} = x/\varepsilon$

b) BL at $x = 0$, $y \sim g(x) - g(0)e^{-\bar{x}}$ where $g(x) = 1 - \int_x^1 f(s)ds$ and $\bar{x} = x/\varepsilon$

c) BL at $x = 0$, $y \sim \varepsilon(1 + 2x + 2e^{-\bar{x}})/3$ where $\bar{x} = x/\varepsilon$

d) BL at $x = 0$, $y \sim \operatorname{sech}(1)[- \sinh(1) - e^x + (1 + e)e^{-\bar{x}}]$ where $\bar{x} = x/\varepsilon$

e) BL at $x = 0$, $y \sim -3x + 4(5 - 3e^{-4\bar{x}})/(5 + 3e^{-4\bar{x}})$ where $\bar{x} = x/\sqrt{\varepsilon}$

g) BL at $x = 0$, $y \sim (1 + 7x^2)^{1/3} - 3\sqrt{2}/[\sinh(z) + \sqrt{2}]$ where $z = \sqrt{3}x/\varepsilon + \operatorname{arcsinh}(2^{3/2})$

2.9

a) $y \sim \beta F(x, 1) - \int_x^1 F(x, r)g(r)dr + e^{-p(0)x/\varepsilon}[\alpha - \beta F(0, 1) + \int_0^1 F(0, r)g(r)dr]$
where $g(r) = f(r)/p(r)$ and $F(x, r) = \exp(\int_x^r q(s)/p(s)ds)$

2.10

a) $y \sim 1 + ae^{\sqrt{x}} - \frac{1+a}{\kappa} \int_{\bar{x}}^{\infty} e^{-s^{3/2}} ds$ where $\bar{x} = x/\varepsilon^{2/3}$, $a = 2/e$, and $\kappa = \frac{2}{3}\Gamma(2/3)$

2.11

a) $y \sim f(x) - f(0) \exp(-\bar{x}^2/2)$ where $\bar{x} = x/\sqrt{\varepsilon}$

b) $y \sim f(x) - f(0) \exp(-q(0)\bar{x}^2/2)$ where $\bar{x} = x/\sqrt{\varepsilon}$

2.22

a) $y \sim \gamma(\gamma-1)^{-1}[f(t) - f(0)e^{-\bar{t}}] + g(\lambda)e^{\bar{t}} + \gamma[g_0 - (\gamma-1)^{-1}f(0)][e^{(1-\gamma)\bar{t}/\gamma} - e^{\bar{t}}]$
 where $\bar{t} = x/\varepsilon$

??

$\partial_t c_0 + f(c_0) = \alpha g(t)$ where $c_0(0) = |\Omega|^{-1} \int_{\Omega} h(\mathbf{x}) dV$ and $\alpha = |\partial\Omega_1|/|\Omega|$

2.13

a) $y \sim x + 12z/(1+z)^2$ where $z = \exp(\sqrt{2}x/\varepsilon)$

Examples Involving Boundary Layers**2.15**

- a) BL at $x = 0, 1$: $y \sim 1 - x - \exp(-x/\sqrt{\varepsilon}) - \exp((x-1)/\sqrt{\varepsilon})$
 b) BL at $x = 0, 1$: $u \sim -f(x) + f(0) \exp(-\bar{x}/\sqrt{E(0)}) + f(1) \exp(\bar{x}/\sqrt{E(1)})$
 where $\bar{x} = x/\sqrt{\varepsilon}$
 c) BL at $x = 1$: $y \sim (2 - e^{2x})/(2 + e^{2x}) + (1 - A)e^{-\bar{x}}$ where $\bar{x} = (x-1)/\varepsilon$
 and $A = (2 - e^2)/(2 + e^2)$
 d) BL at $x = 0, 1$: $y \sim -e^x f(x) + (f(0) + 1)e^{-\bar{x}} + (ef(1) - 1)e^{\bar{x}\sqrt{e}}$
 e) BL at $x = 0, 1$: $y_0 = \sqrt{1 + x^2}$
 g) BL at $x = 0, 1$: $y \sim e^x - e^{-\bar{x}} + (4 - e)e^{\bar{x}}$ where $\bar{x} = x/\varepsilon^{5/4}$
 h) BL at $x = 1$: $y \sim \sqrt{7} - \sqrt{9 - 2x} + Y_0(\bar{x})$ where

$$\int_{-2}^{Y_0} \frac{dr}{A - r^2/2 + r^3/3} = \bar{x}$$

for $A = (-10 + 7\sqrt{7})/3$

- i) BL at $x = 1$
 j) BL at $x = 0, 1$: $y \sim -x + 2e^{-4\bar{x}} + 4e^{r\bar{x}}$ where $r = -1 + \sqrt{5}$, $\bar{x} = x/\varepsilon$, and
 $\bar{x} = (x-1)/\varepsilon$
 k) BL at $x = 1$: $y \sim Y_0(\bar{x})$ where $\bar{x} = (x-1)/\sqrt{\varepsilon}$, and

$$\int_{Y_0}^2 \frac{dr}{\sqrt{2(r - \ln(r) - 1)}} = -\bar{x}$$

- l) BL at $x = 0, 1$: $y \sim e^x + Y_0(\bar{x}) - 1 + Z_0(\bar{x}) - e$ where

$$\int_2^{Y_0} \frac{dr}{r\sqrt{2(r - \ln(r) - 1)}} = -\bar{x}$$

- m) BL at $x = 1$: $y \sim y_0(x) + Y_0(\bar{x}) - \alpha$ where $\frac{1}{3}y_0^3 - y_0 = 6 - kx$ for $y_0(0) = 3$,
 $\alpha = y_0(1)$, $\bar{x} = (x-1)/\varepsilon$, and

$$\int_2^{Y_0} \frac{4dr}{(r^2 - \alpha^2)(r^2 + \alpha^2 + 2)} = \bar{x}$$

2.17

BL at $x = 0, 1$: $y \sim \kappa_0^{-1}(1 - e^{-(x+1)/\varepsilon} - e^{(x-1)/\varepsilon})$ and $\kappa \sim \kappa_0 + \dots \Rightarrow \kappa_0 = 2^{1/3}$

2.18

a) $y \sim H(0)/H(x) + Y_0(\bar{x}) - H(0)/H(1)$ and $\bar{x} = (x - 1)/\varepsilon$ and

$$\bar{x}/H(1) = H(1)(Y_0 - 1) + H(0) \ln \left| \frac{H(0) - H(1)Y_0}{H(0) - H(1)} \right|$$

2.19

iii) $y \sim \arctan \frac{1}{1-x} + 4\arctan(ae^{-\alpha\bar{x}})$, where $a = \tan(\pi/16)$

2.23

b) $\lambda \sim n^2\pi^2(1 + 4\varepsilon + (12 + n^2\pi^2)\varepsilon^2)$

2.24

$y \sim -f(x)/q(x) + (\alpha + f_0/q_0)e^{r_0x/\varepsilon} + (\beta + f_1/q - 1)e^{r_1(x-1)/\varepsilon}$ where $f_0 = f(0)$, etc and $r_0 = -p_0 - \sqrt{p_0^2 + q_0}$ and $r_1 = -p_1 + \sqrt{p_1^2 + q_1}$

2.25

a) $\gamma = (\beta - \alpha)/2$

b) $\phi = \varepsilon\alpha + \sqrt{\varepsilon}\gamma\Phi(x, \varepsilon)$

c) outer: $\phi \sim \varepsilon\alpha^{1/(2k+1)}$, BL at $x = 0$, $\Phi \sim -\varepsilon^\kappa/(B + k\bar{x}/\sqrt{k+1})^{1/k}$, where $\kappa = (2k+1)/(2k+2)$, $B = (-\gamma\sqrt{k+1})^{-k/(k+1)}$ and $\bar{x} = x/\varepsilon^\kappa$; solution symmetric about $x = 1/2$

d) it's necessary to find the second term in the BL to be able to match

e) $\gamma = (\beta - \alpha|\Omega|)/|\partial\Omega|$

Interior Layers**2.32**

a) IL at $x_0 = 1/2$: $y \sim \operatorname{erf}((x - x_0)/\sqrt{2\varepsilon})$

b) IL at $x_0 = 0$

c) IL at $x_0 = 7/12$: for $0 \leq x \leq x_0$, $y \sim 3/(5 - 3x) - 2\beta/(1 + Be^{-\beta\bar{x}})$ and for $x_0 \leq x \leq 1$, $y \sim -2/(1 + 2x) + 2\beta Be^{-\beta\bar{x}}/(1 + Be^{-\beta\bar{x}})$ where $\beta = 12/13$, $\bar{x} = (x - x_0)/\varepsilon$ and $B > 0$ is constant.

d) IL at $x_0 = 0$

e) IL at $x_0 = 1/2$: for $x_0 \leq x \leq 1$, $y \sim y_0(x) + Y_0(\bar{x}) - \alpha$ where $y_0 + y_0^3/3 = x/2 - 4/3$, $\alpha = y_0(x_0^+)$, $\bar{x} = (x - x_0)/\varepsilon$, and

$$\int_0^{Y_0} \frac{4dr}{(\alpha^2 - r^2)(r^2 + \alpha^2 + 2)} = \bar{x}$$

f) IL at $x_0 = 2/3$: for $0 \leq x \leq x_0$, $y \sim -1 - 3x - 6Be^{3\bar{x}}/(1 + Be^{3\bar{x}})$ where $\bar{x} = (x - x_0)/\varepsilon$ and $B > 0$ is constant.

OLD b) IL at $x_0 = 0$: $y \sim -2/x$ for $x \neq 0$, and $Y \sim 4\bar{x}M(1, 3/2, -\bar{x}^2)/\sqrt{\varepsilon}$ where $\bar{x} = x/\sqrt{\varepsilon}$

OLD d) BL at $x = 0$ and IL at $x_0 = 3/4$, for $0 \leq x \leq x_0$, $y \sim Y_0 - 3 \exp(-3\bar{x}/16)$ and for $x_0 \leq x \leq 1$, $y \sim Y_0 + [\sqrt{x - 1/4} - 1/\sqrt{2}]/(x - x_0)^{3/2}$ where $Y_0 = 1 + \kappa[\Gamma(-1/4)M(3/4, 1/2, -\tilde{x}^2/4) + \Gamma(1/4)M(5/4, 3/2, -\tilde{x}^2/4)]$. Also, $\kappa^{-1} = 32\varepsilon^{3/4}\sqrt{3\pi}$, $\tilde{x} = (x - x_0)/\sqrt{\varepsilon}$, and $\bar{x} = x/\varepsilon$

XXXXXXXX I think a \tilde{x} is missing in IL sol XXXXXXXXXXXX

2.35

- a) $y \sim 1 + \sqrt{1 - x^2}$
- c) $y \sim 1 + \sqrt{1 - x^2}$ for $0 \leq x < x_0 = \sqrt{3}/2$, $y \sim 0$ for $x_0 < x < 1$, and in the IL at x_0 , $y \sim Z_0(\tilde{x})$ where $2/Z_0 + (4/3) \ln((3/2 - Z_0)/Z_0) = \tilde{x} + B$ for $\tilde{x} = (x - x_0)/\varepsilon$
- d) in the BL at $x = 1$, $y \sim Y_0(\bar{x})$, where $(2 - 3/Y_0) \exp(3/(2Y_0)) = (7/2) \exp(3(\bar{x} - 1)/4)$

2.37

b) BL at $x = 0, 1$: $y \sim k(2x - 1) + (-3 - k)e^{\tilde{x}} - ke^{-\tilde{x}}$, where $\tilde{x} = (x - 1)/\varepsilon$ and $\bar{x} = x/\varepsilon$

2.38

a) outer: $y \sim \frac{1}{2}(k + \sqrt{k^2 + 4\alpha})$ for $k = 1 - \alpha - x/2$; BL: $\bar{x} = (x - 1)/\varepsilon$, $4a = 1 - 2\alpha + \sqrt{1 + 12\alpha + 4\alpha^2}$, $4b = -1 + 2\alpha + \sqrt{1 + 12\alpha + 4\alpha^2}$, and $0 \leq Y \leq a$ satisfies

$$\left(\frac{a - Y}{a}\right)^a \left(\frac{b + Y}{b}\right)^b = e^{2(a+b)\bar{x}}$$

2.41

- a) $T = T_\infty$
- b) $T \sim 1 - \varepsilon \ln[1 - (T_\infty - 1)^nt]$
- c) $\mu = \varepsilon^{-n} \exp((1 - T_\infty)/(\varepsilon T_\infty))$ and

$$\tau + \tau_0 = - \int_1^{T_1} r^{-n} \exp(r/T_\infty^2) dr$$

2.40

c) $y = -2 + x/4 + c \int_0^x \exp(-4g(s)/\varepsilon) ds$, where $g(x) = x^3/3 - (a+b)x^2/2 + abx$

and c is an ε dependent constant that makes $y(1) = 3$. The desired conclusion comes from the fact that $c \Rightarrow 0$ if $g(x) < 0$ anywhere in the interval $0 < x < 1$.

Corner Layers

2.43

a) CL at $x_0 = \frac{1}{2}$: $y \sim x_0 + |x - x_0|$ and $Y \sim x_0 + \varepsilon[2 \ln(1 + e^{\bar{x}}) - \bar{x}]$ where $\bar{x} = (x - x_0)/\varepsilon$

b) CL at $x_0 = \frac{3}{4}$

c) CL at $x_0 = \frac{1}{3}$: $y \sim -1 + 3|x - x_0|$

d) CL at $x_0 = 0$: in the CL, $Y \sim \sqrt{\varepsilon} [\frac{1}{2}\bar{x}(-3 + \operatorname{erf}(\bar{x})) - \exp(-\bar{x}^2)/(2\sqrt{2\pi})]$ where $\bar{x} = x/\sqrt{\varepsilon}$

OLD c) CL at $x_0 = 0$: $y_r \sim -xe^{-x}$ and $y_l \sim x(2e - e^{-x})$

OLD d) BL at $x = 0$ and CL at $x_0 = 5/8$: $y_r \sim 0$ for $0 < x \leq x_0$, and $y_l = 1 - \sqrt{9/4 - 2x}$ for $x_0 \leq x \leq 1$

OLD e) BL at $x = 0$ and CL at $x_0 = 1/2$: $y_r \sim 1/2$ for $0 < x \leq x_0$ and $y_l \sim x$ for $x_0 \leq x \leq 1$

2.47

for $0 \leq x < a$:

$$y \sim 1 - 3 \left(\frac{b}{b-x} \right)^{\frac{b}{b-a}} \left(\frac{a-x}{a} \right)^{\frac{a}{b-a}}$$

for CL at $x = a$:

$$Y \sim 1 + \varepsilon^\gamma \alpha_0 [M(-\gamma, 1/2, -(b-a)\bar{x}^2/2) - \kappa \bar{x} M(1/2 - \gamma, 3/2, -(b-a)\bar{x}/2)]$$

where $\gamma = a/(2(b-a))$ and $\kappa = \sqrt{2(b-a)}\Gamma(\eta + 1/2)/\Gamma(\eta)$ for $\eta = b\gamma/a$

2.48

Outer: $y \sim y_0(x)$ where

$$\int_{y_0}^2 \frac{dr}{\operatorname{arctanh}(r-1)} = 1 - x$$

BLs at $x = 0, 1$

CL: $\tilde{x} = (x - x_0(\varepsilon))/\varepsilon$ where $x_0 \sim \sqrt{\varepsilon} \ln(3/y_0(0))$, $y_0(0) \approx 1.3$, and $\tilde{Y} \sim y_0(0) + \varepsilon \tilde{Y}'_0$ for $\tilde{Y}'_0 = \operatorname{arctanh}(T)$ where $-1 < T < \lambda$ and satisfies

$$\frac{(\lambda - T)^2}{(1 + T)^{1-\lambda}(1 - T)^{1+\lambda}} = A e^{2(\lambda^2 - 1)\tilde{x}}$$

where A is positive and $\lambda = y_0(0) - 1$.

2.49

- a) BLs at $x = 0, 1$ and a CL at $x = 1/3$
 b) ILs at $x = \frac{1}{3}\sqrt{2} - \frac{3}{8}$ and at $x = \frac{11}{8} - \frac{1}{3}\sqrt{2}$
 c) IL at $x = \frac{1}{3}\sqrt{2} - \frac{3}{8}$, a CL at $x = 1/3$ and a BL at $x = 1$

PDEs

2.55

- a) $u \sim h(x+s) + [g(t) - h(s)] \exp(-xs(t)/\varepsilon]$ where $s = \int_0^t v(\tau) d\tau$
 b) the second term is $u_1(x, t) = th''(x+s)$

2.58

- b) $X \sim s(t) + 2\varepsilon \ln(B)/(u_0^+ - u_0^-)$

2.59

- a) outer layers ($x \neq \alpha t$): $u \sim \phi(x - \alpha t) \exp(-\beta t)$
 transition layer: $U \sim e^{-\beta t} [\phi(0^+) + \frac{1}{2}(\phi(0^-) - \phi(0^+)) \operatorname{erfc}(z)]$ where $z = (x - \alpha t)/(2\sqrt{\varepsilon t})$

2.60

- outer layers ($x \neq s(t)$): $u(x, t) \sim \phi(x_0)$ where $x = x_0 + f(\phi(x_0))t$ and $x_0 \neq 0$
 shock layer: letting $F'(u) = f(u)$, where $F(u_0^+) = 0$, and $G(u) = F(u)/(u - u_0^+)$, then $s'(t) = G(u_0^-)$ and

$$\int_{u_0^+}^{U_0} \frac{dr}{(r - u_0^+)[G(r) - G(u_0^-)]} = \bar{x} + B$$

2.61

$$w \sim -\kappa e^{-\mu t} \int_0^t e^{(-2+\mu)\tau} \operatorname{erf}((1-r)/\sqrt{2\varepsilon(t-\tau)}) d\tau$$

2.63

$$c \sim \frac{1}{r+\mu} \operatorname{erfc}(x/(2\sqrt{t})) + (1 - \frac{1}{r+\mu}) \operatorname{erfc}(\frac{x}{2\varepsilon} \sqrt{\frac{r+\mu}{\mu t}})$$

Difference Equations

2.67

- a) $y_n \sim Ar_+^n + Br_-^n$ where $r_{\pm} = (\beta \pm \sqrt{\beta^2 + \alpha^2})/\alpha$

b) $\alpha = hp_n/2$, $\beta = h^2q_n/2$; the second derivative does not contribute

Chapter 3 Multiple Scales

Introductory Example

3.1

a) $y \sim 2 \sin(t)/\sqrt{4 + 3\varepsilon t}$

d) $y \sim \sqrt{\varepsilon} \exp(-\alpha t/2) \sin(t/\sqrt{\varepsilon})$

3.2

$\theta \sim \varepsilon \cos((1 - \varepsilon^2/16)t)$

3.13

a) $y \sim A(\tau) \cos(t) + B(\tau) \sin(t)$, where $A(0) = 0$, $B(0) = 1$,

$$A' = \frac{1}{2\pi} \int_0^{2\pi} F(-A \sin(t) + B \cos(t)) \sin(t) dt$$

and

$$B' = -\frac{1}{2\pi} \int_0^{2\pi} F(-A \sin(t) + B \cos(t)) \cos(t) dt$$

b) $A = 0$ and $B = 3\pi/(3\pi + 4\tau)$

3.6

$\phi \sim \alpha [1 - \sqrt{2} \exp(-(1 + \gamma)\varepsilon t/2) \sin(t + \pi/4)]$

3.8

a) $\phi \sim A(x, t) \exp(-iv(x) \sin(t/\varepsilon))$, where $2iA_t + 2A_{xx} = v_x^2 A$.

3.14

b) $\mathbf{q} = \mathbf{q}_0 \exp(-i\omega t_1)$, where $\omega = \sqrt{\beta\gamma - \alpha^2}$ and $\mathbf{q}_0 = (\beta, i\omega - \alpha)^T$

c) $\mathbf{p} = \alpha_0 \mathbf{p}_0 \exp(i\omega t_1) + cc$, where $\mathbf{p}_0 = (\gamma, i\omega + \alpha)^T$

3.11

b)

$$\alpha_j = \frac{1}{2^{2m+1}} \left[\binom{2m+1}{m+1-j} + \binom{2m+1}{m+j} \right]$$

c) If $j \neq 0$ then

$$\gamma_j = \frac{1}{2^{2m}} \left[\binom{2m}{m+j} + \binom{2m}{m-j} \right]$$

and

$$\gamma_0 = \frac{1}{2^{2m}} \binom{2m}{m}$$

Weakly Coupled Oscillators

3.30

c)

$$\bar{\kappa}_1 = \frac{1}{2m_1} \frac{k_0^2 c_0}{k_0^2 + c_0^2 \omega_1^2}$$

$$\bar{\theta}_1 = \omega_1 \left[1 + \frac{1}{2m_1} \frac{\varepsilon k_0 c_0^2}{k_0^2 + c_0^2 \omega_1^2} \right]$$

where $\omega_1^2 = k_1/m_1$.

Slowly Varying Coefficients

3.35

$y \sim D(\varepsilon t)^{-1/4} [\beta D(0)^{3/4} \sin(\tau) + \alpha D(0)^{1/4} \cos(\tau)]$, where $\tau = \int_0^t D(\varepsilon s)^{-1/2} ds$

3.37

The higher order terms for the stability boundaries can be found in Abramowitz and Stegun (1972), pg 724. Note, using their notation, $a = 4\lambda$ and $q = \varepsilon/4$

3.41

b) $s = g(x)/g(1)$, where $g(x) = \int_0^x \sqrt{1 + \varepsilon\mu(r)} dr$, where $\nu = g(1)$

c) $h \sim \mu'(s)/2 \Rightarrow \nu \sim n\pi - \varepsilon v_1$, where $4v_1 = \int_0^1 \mu'(s) \sin(2n\pi s) ds$

d) $v_1 = 0 \Rightarrow \lambda \sim \lambda_0 + \varepsilon\lambda_1$, where $\lambda_0 = n\pi$, and $\lambda_1 = v_1 - v_0/2 = -n\pi/2$.

Also, $Y \sim A(1 - \varepsilon s/4) \sin(n\pi s) + B\varepsilon \sin(n\pi s)$ and $s \sim x[1 + \varepsilon(x-1)/4]$

e) the eigenfunctions have the form $y_n = \sqrt{1 + \varepsilon x} \sin(\alpha_n \ln(1 + \varepsilon x))$, where α_n is an ε dependent constant

Forced Motion Near Resonance

3.18

a) $u \sim \alpha + (1 - \alpha) \cos(\theta - \varepsilon\alpha^2\theta)$

b) $2\pi(1 + \alpha^2\varepsilon)$

c) $\Delta\phi \approx 42.98''$

3.19

a) $y \sim A(\varepsilon t) \cos(t + \theta_0)$, where $A = 2$ or $A = 2\sqrt{c/(c + 4 \exp(-\varepsilon t))}$

3.20

b)

$$y_0 = -\frac{1}{16} \cos(3t_1 + \omega t_2) + 2r(t_2) \cos(t_1 + \theta(t_2))$$

where

$$2r' = -\frac{3}{16} \kappa r^2 \sin(\theta - \omega t_2)$$

and

$$2\theta' = \kappa \left(\frac{3}{128} - \frac{3}{16} r^2 \cos(\theta - \omega t_2) + 3r^2 \right)$$

where $r(0) = 1/16$ and $\theta(0) = 0$.

b) $A_\infty^2 [9\lambda^2 + (6\omega - \frac{3\kappa}{4} A_\infty^2)^2] = 1 \Rightarrow A_M = 1/(3\lambda)$

3.21

a) $\theta \sim \varepsilon^2 (\sin(\omega t) - \omega \sin(t)) / (1 - \omega^2) + O(\varepsilon^8)$

b) $2A' = -\cos(\theta - \omega \varepsilon t)$ and $A_\infty^3 + 16\omega A_\infty \pm 8 = 0$

3.22

$y \sim \varepsilon^{1/3} A(\varepsilon t) \cos[\kappa t + \theta(\varepsilon t)]$, where $A_\infty (\frac{3\lambda}{8} A_\infty^2 - 2\omega \kappa) = 1$

3.23

a) $y \sim \varepsilon^{1/3} A(\varepsilon^{2/3} t) \cos[t + \theta(\varepsilon^{2/3} t)]$, where $A_\infty (\alpha^2 + (2\omega + A_\infty^2)^2) = 1$

b) yes

3.24

a) $y \sim f(v)(1 - e^{-\gamma \varepsilon t} \cos(t))$, where $2\gamma = \beta + a(v^2 - 1)$

b) $y \sim \frac{1}{\varepsilon} A(\varepsilon t) \cos(t + \theta_0)$, where $2A' + [\beta + a(v^2 - 1 + \frac{1}{4} A^2)]A = 0$

Other references: Derjaguin, et al. (1957) and Vatta (1979)

3.27

a) $\theta \sim \varepsilon^{1/3} A(\varepsilon t) \cos[t + \phi(\varepsilon t)]$, where $8A' = [\alpha \sin(2\phi) - 4\mu]A$, and if $A \neq 0$ then $16\phi' = 2\alpha \cos(2\phi) - A^2 - 4\alpha$. Note that $\alpha < 4\mu \Rightarrow A \rightarrow 0$

Introduction to Partial Differential Equations**3.48**

a) $u(x, t) \sim \sum_{n=1}^{\infty} \beta_n e^{-\varepsilon t/2} \cos((\lambda_n^2 - \varepsilon/2)t) \sin(\lambda_n x)$

3.50

b) $u \sim \varepsilon(1-x)U(\tau)$ and $\rho \sim 1/[1 - \int_0^\tau U(\sigma)d\sigma]$

Linear Wave Propagation**3.53**

b) $u \sim \sqrt{\frac{c(0)}{c(\varepsilon t)}} [f(x - \int_0^t c(\varepsilon\tau)d\tau) + f(x + \int_0^t c(\varepsilon\tau)d\tau)]/2$

3.54

b) $u \sim \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} [f(x-t+2r\sqrt{\varepsilon t})e^{-r^2}] dr$

3.55

a) $p \sim [F(x-t) + G(x+t)]\sqrt{\frac{A(0)}{A(\varepsilon x)}}$

3.56

characteristics satisfy $x_t = y$, $y_t = -f(t)x \Rightarrow x'' + (a - \varepsilon \cos(t))x = 0$ which is Mathieus' eq

Nonlinear Waves**3.57**

$u(x, t) \sim \alpha \cos\left[kx - \left(\omega - \frac{5\alpha^2\varepsilon^2}{12\omega\lambda}\right)t\right]$

3.60

b) The solution has the form given in (a), except that

$$F = \frac{p(\theta_1)}{\gamma p(\theta_1)[1 - \exp(t_2/(1+c))] + 2 \exp(t_2/(1+c))}$$

and

$$G = \frac{q(\theta_2)}{\gamma q(\theta_2)[1 - \exp(t_2/(1+c))] + 2 \exp(t_2/(1+c))}$$

where $p = g + \sqrt{2}h$ and $q = g - \sqrt{2}h$, and $u(x, 0) = g(x)$ and $v(x, 0) = h(x)$

3.61

b) $u \sim \varepsilon\{B_0 + A_0 \cos[(\theta + \phi(\tau))]\}$, where $\theta = kx - \omega t$ and $\tau = \varepsilon^\gamma t$; i) $B_0 \neq 0 \Rightarrow \gamma = 1$, $\phi = \phi_0 - \alpha\kappa\tau B_0$, and ii) $B_0 = 0 \Rightarrow \gamma = 2$, $\phi = \phi_0 + C\tau$ for $C = \alpha k c_0 - \alpha^2 A_0^2 / (24\beta k)$ where c_0 is an arbitrary constant

c) $u \sim \varepsilon[\mu + A(r) \cos[\theta + \alpha\varepsilon\mu(x-t)/(3\beta k) + \phi_0(r)]]$, where $r = \varepsilon x - (1 - 3\beta k^2)\varepsilon t$; the case of when $\mu = 0$ is considered in Schoombie (1992)

3.58

b) $u = 1/(1 + \exp(\lambda(x + t) - \varepsilon t))$

3.65

b) $u_0 = U_0(t_1 + \theta(x, t_2))$, where $U_0(s) = 1/(1 + e^{-s})$ and $\theta = \frac{1}{2} \ln(1 + 4\varepsilon\lambda t) - [\varepsilon\lambda^2 t(x_1 - x_0)^2 - \lambda(x - x_0)(x - x_1)]/(1 + 4\varepsilon\lambda t)$

3.67

b) $u = u_0(\theta, x_2)$, for $\theta = t - x$ and $x_2 = \varepsilon x$, where $u_0 = 0$ if $\theta < 0$ otherwise $\partial_\theta u_0 = g'(s)$ where $s = s(\theta, x_2)$ is the non-negative solution of $\theta = s - \frac{1}{2}x_2 f(-g'(s))$. Thus, for $\theta \geq 0$, $u_0 = g(s) + \frac{1}{2}x_2[g'(s)f(-g'(s)) + F(-g'(s))]$ where $F'(\sigma) = f(\sigma)$ with $F(0) = 0$.

3.68

b) $s \sim \frac{1}{2}[F(\phi_1) + F(\phi_2)]$, where $\phi_1 = x - t + \varepsilon t F(\phi_1)/2$ and $\phi_2 = x + t + \varepsilon t F(\phi_2)/2$

3.69

a) $\rho \sim 1 + \varepsilon f(x - \lambda t) + 2\varepsilon^2 \lambda t f'(x - \lambda t)$ where $\lambda = \alpha + \beta$

Difference Equations**3.72**

a) $y_n \sim A e^{2\varepsilon n}$

3.77

a) $y_n \sim \alpha^n \bar{y}_0(s)$, where $\bar{y}'_0 = g(\bar{y}_0)$

3.78

from (a) $\theta' = \frac{3}{8}A^2$; from (i), $\theta' \sim \frac{3}{8}A^2(1 + \frac{1}{8}h^2)$; from (ii), $\theta' \sim \frac{3}{8}A^2(1 - \frac{5}{8}h^2)$; from (iii), $\theta' \sim \frac{3}{8}A^2(1 - \frac{1}{8}h^2)$

3.79

a) $\omega^2 = \omega_d^2 + 4 \sin^2(k/2)$

b) $u_n(t) \sim \varepsilon A(\bar{t}) \cos(kn - \omega t + \phi(\bar{t}))$

3.80

a) $g_n \sim G_0(\varepsilon n)$, where $G'_0(s) = \gamma G_0(1 - G_0)(\alpha + \beta G_0)$, where $\gamma = \frac{1}{N} \sum_{j=1}^N f(j)$

b) $g_n \sim 0, 1, -\alpha/\beta$; $0 < g_\infty < 1$ if $\alpha > 0$ and $\beta < 0$ with $0 < g_1 < 1$

Chapter 4

The WKB and Related Methods

Introductory Example

4.1

c) $y_{WKB} \sim A[\exp(-e^x) - \exp(e^x - 2 - x/\varepsilon)]$, where $A = 1/[\exp(-e) - \exp(e - 2 - 1/\varepsilon)]$, and $y_c \sim \exp(e - e^x) - \exp(-1 + e - x/\varepsilon)$

4.3

d) $y \sim q^{-1/4}(Ae^{\eta/\varepsilon} + Be^{-\eta/\varepsilon}) \exp(-\frac{1}{2} \int^x f(r) dr)$, where $\eta = \int^x \sqrt{q(r)} dr$

4.9

c) $y \sim h^{-1/2}[a_0 \exp(\frac{1}{2\varepsilon} \int^x (-p + h + \frac{\varepsilon p'}{h}) ds) + b_0 \exp(-\frac{1}{2\varepsilon} \int^x (p + h + \frac{\varepsilon p'}{h}) ds)]$, where $h = \sqrt{p^2 - 4q}$

4.5

c)

$$w \sim g(x)[(a + b/n) \sin(\lambda_n \theta) - \frac{\arccos(\lambda_n \theta)}{2\lambda_n} \int_0^x \frac{1}{\sqrt{pq}} (f - 2q\lambda_0\lambda_2 - pg''/g) ds].$$

where a and b are constants, $\theta = \int_0^x \sqrt{q/p} ds$ and $\lambda_0 = \pi/\kappa$. [XXX why λ_2 here? XXX]

$$e) \lambda \sim n \left(1 - \frac{49}{72(n\pi)^2}\right)$$

4.6

c) $y_\pm \sim q^{-1/4} \exp(\frac{1}{2\varepsilon} \int^x (\pm \sqrt{-4q + \varepsilon p^2} - \sqrt{\varepsilon p}) dx)$.

4.8

a) $y_\pm \sim \kappa_\pm x^{\pm\nu} [1 \mp \frac{1}{4\nu}(\alpha_\pm + x^2) + \dots]$.

b) $J_\nu(x) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{\varepsilon x}{2\nu}\right)^\nu [1 - \frac{1}{12\nu}(1 + 3x^2)]$.

4.13

a) $y \sim \varepsilon e^{t/2} \{ \sin[\alpha(1 - e^{-t})] + \frac{1}{8}\varepsilon(e^t - 1) \cos[\alpha(1 - e^{-t})] \}$, where $\alpha = 1/\varepsilon$

b) two principal shortcomings are: (1) differences in zeros — this is a relatively minor problem as discussed in Section 1.4, and (2) unboundedness of second term for large t — in particular, we need $\varepsilon e^t \ll 1$. The latter problem is due to a turning point at $t = \infty$.

c) $y = \frac{\pi}{2} [Y_0(\alpha)J_0(\beta) - J_0(\alpha)Y_0(\beta)]$ where $\alpha = 1/\varepsilon$ and $\beta = \alpha e^{-t}$

4.14

a) $\varepsilon\beta R'' + (-1 + \lambda\beta)R' = 0$ and $\varepsilon\beta R'(0) - R(0) = 0 \Rightarrow R = 1 - re^{-kx/\varepsilon}$

where $r = 1/(\beta\lambda)$ and $k = (-1 + \lambda\beta)/\beta$

b) $R \sim 1 + Ae^{-\theta(x)/\varepsilon}/(1 - \lambda\beta)$ where $\theta = \lambda x - \int_0^x \frac{ds}{\beta(s)}$ and $A = 1 - 1/(\lambda\beta_0)$; the conditions are that $\theta \rightarrow \infty$ as $x \rightarrow \infty$ and $\lambda\beta > 0$ for $0 < x < \infty$

4.15

$P \sim A(x, \varepsilon) \exp(\theta(x)/\varepsilon + \eta\mu_0 x(\exp(\theta_x) - 1) - \int_0^\eta \lambda(s) ds)$, where $\theta_x = \ln(\xi)$ and ξ satisfies $\mu_0 x \int_0^\infty \exp\left(\mu_0 x(\xi - 1)\eta - \int_0^\eta \lambda(s) ds\right) d\eta = 1$

Turning Points

4.18

b) $E = \varepsilon(2n + 1)\sqrt{\mu}$

d) $E \sim (\alpha\mu\varepsilon^m N^2)^{1/\gamma}(1 + \beta/N^2)$, where $4\alpha = \pi \left[\Gamma\left(\frac{3m+2}{2m}\right)/\Gamma\left(\frac{m+1}{m}\right)\right]^2$ and $3\pi(m+2)^2\beta = 4m(m-1)\cot(\pi/m)$

4.19

b) the approximation for $\Psi(x)$ is given by (4.48) for $x < a$, and for $x > b$ one finds $\Psi \sim 2a_R|q(x)|^{-1/4}e^{-\phi/\varepsilon} \exp(i(-Et - \bar{\theta}/\varepsilon - \pi/4))$ where $q = V(x) - E$, $\phi = \int_a^b \sqrt{q(s)} ds$ and $\bar{\theta} = \int_x^b \sqrt{q(s)} ds$

4.23

b) balancing $\Rightarrow \alpha = 1/2$ and $\beta = 1/4$

Wave Propagation and Energy Methods

4.30

b) $E = \frac{1}{2}Du_{xx}^2 + \frac{1}{2}\mu u_t^2$, $S = -u_{xt}Du_{xx} + u_t\partial_x(Du_x)$, $\Phi = 0$

Wave Propagation and Slender Body Approximations

4.35

a) for the n th mode the general solution has the same form as in (4.102) but

$$u_L = \frac{\sin(\lambda_n y)}{\sqrt{\theta_x}} \left(a_L e^{i(\omega t - \theta(x)/\varepsilon) + \zeta} + b_L e^{i(\omega t + \theta(x)/\varepsilon) - \zeta} \right)$$

where $\theta(x)$ is given in (4.105) and $\zeta(x) = \frac{\omega}{2} \int_0^x \alpha(x)/\sqrt{\omega^2\mu^2 - \lambda_n} dx$

Ray Methods

4.52

a) $\mathbf{k} = \omega\mu\mathbf{u}$, where \mathbf{u} is the unit vector given in the ray equation initial condition.

b) Given that $\theta_I = \theta_R$ on the interface it follows that $(\mathbf{u}_I - \mathbf{u}_R) \cdot \mathbf{x} = 0, \forall \mathbf{x} \in S$. Consequently, $\mathbf{u}_I - \mathbf{u}_R$ is parallel to \mathbf{n} , and from this it follows that the three

vectors are coplanar. That the angles are equal follows from the identity $\mathbf{u} \cdot \mathbf{n} = \cos(\varphi)$.

4.48

a) $D(\nabla\theta \cdot \nabla\theta)^2 = \mu$.

4.43

b) SIAM Appl Math, v16, 1968, 783-807

??

a) $\alpha = \mu(-1/2) \sin(\phi)$ where $z_R = \min\{0, \text{roots of } \mu(z) = \alpha \text{ for } z > -1/2\}$

c) $v_0 = \sqrt{\tan(\phi)/(\mu J)/(4\pi)}$

4.50

a) $\mu = \text{constant} \Rightarrow$ rays are straight lines \Rightarrow wave fronts parallel $\Rightarrow R_i = \rho_i + s$; in \mathbb{R}^2 , $\rho_2 = \infty$

4.45

a) (4.134) $\Rightarrow \bar{\mathbf{x}}_{ss} = a\bar{\mathbf{x}} - b\bar{\mathbf{x}}_s \Rightarrow \mathbf{p}_s = \mathbf{0}$

b) $\kappa = \pm r_0 \mu(r_0) \sin(\phi_0)$ where ϕ_0 is the initial angle the ray makes with the radius vector

4.51

c) note $\kappa \approx 0.065r_c$ where r_c is the scale factor (in km) used to nondimensionalize $r^* = r_c r$

4.44

Claim 1: $\partial_s \mathbf{X}|_{s=0} = \kappa \mathbf{n}$, for some scalar κ .

The first thing to note, from the WKB expansion in (4.120) and the ray equation (4.124),

$$\begin{aligned} \nabla v &\sim e^{i\omega\theta(\mathbf{x})} [i\omega v_0 \nabla\theta + \dots] \\ &= e^{i\omega\theta(\mathbf{x})} \left[\frac{i\omega}{\lambda} v_0 \partial_s \mathbf{X} + \dots \right]. \end{aligned}$$

In what follows it is assumed that the boundary surface S is parameterized as $\mathbf{x} = \mathbf{r}(\alpha, \beta)$, where the tangent vectors $\mathbf{t}_\alpha = \partial_\alpha \mathbf{r}$ and $\mathbf{t}_\beta = \partial_\beta \mathbf{r}$ are orthonormal. If \mathbf{n} is the unit outward normal to S then, using this and the tangent vectors as a basis, we can write

$$\nabla v = (\nabla v \cdot \mathbf{n}) \mathbf{n} + (\nabla v \cdot \mathbf{t}_\alpha) \mathbf{t}_\alpha + (\nabla v \cdot \mathbf{t}_\beta) \mathbf{t}_\beta.$$

Also, from the boundary condition (4.119), it follows that on S ,

$$\partial_\alpha v = \nabla v \cdot \partial_\alpha \mathbf{X} = \nabla f \cdot \mathbf{t}_\alpha,$$

and

$$\partial_\beta v = \nabla v \cdot \partial_\beta \mathbf{X} = \nabla f \cdot \mathbf{t}_\beta.$$

Combining our results, we have shown that

$$(\nabla v \cdot \mathbf{n})\mathbf{n} + (\nabla f \cdot \mathbf{t}_\alpha)\mathbf{t}_\alpha + (\nabla f \cdot \mathbf{t}_\beta)\mathbf{t}_\beta \sim \frac{i\omega}{\lambda} f(\mathbf{x}_0) \partial_s \mathbf{X}|_{s=0} + \cdots \quad (\text{F.1})$$

This assumes that

$$\theta|_{s=0} = 0. \quad (\text{F.2})$$

Given that S and f are independent of ω , and assuming that λ is independent of ω , (F.1) shows that $\partial_s \mathbf{X}|_{s=0}$ is a scalar multiple of \mathbf{n} . In other words, $\partial_s \mathbf{X}|_{s=0} = \kappa \mathbf{n}$.

Claim 2: $\kappa = \lambda\mu$

To find κ , note that from (4.122) and (4.124),

$$\frac{\partial \mathbf{X}}{\partial s} \cdot \frac{\partial \mathbf{X}}{\partial s} = \lambda^2 \mu^2. \quad (\text{F.3})$$

From this, it follows that $\kappa^2 = \lambda^2 \mu^2$.

Therefore,

$$\left. \frac{\partial \mathbf{X}}{\partial s} \right|_{s=0} = \lambda_0 \mu_0 \mathbf{n}_0, \quad (\text{F.4})$$

where \mathbf{n}_0 is the unit outward normal at \mathbf{x}_0 , and λ_0 and μ are the values of the respective quantities at \mathbf{x}_0 . This, along with (4.130), are the initial conditions used in solving (4.129).

Discrete WKB Method

4.62

a) $y_n \sim (q_n^2 - 4)^{-1/4} [a_0 \exp(\theta_+/\varepsilon) + b_0 \exp(\theta_-/\varepsilon)]$ where

$$\theta_\pm = \int^{\varepsilon n} \ln \left(\frac{1}{2} (q \pm \sqrt{q^2 - 4}) \right) d\nu$$

b) $\alpha_{n+1} = c_n \alpha_{n-1} / a_n$

4.63

a) $y_n \sim [a_0 \exp(i\theta/\varepsilon) + b_0 \exp(-i\theta/\varepsilon)] / (1 - \nu^2)^{1/4}$ where $\nu = \varepsilon n$ and $\theta = \nu \arccos(\nu) - \sqrt{1 - \nu^2}$

b) $a_0 = \sqrt{\varepsilon/(2\pi)} e^{i\pi/4}$ and $b_0 = \sqrt{\varepsilon/(2\pi)} e^{-i\pi/4}$

4.64

a) $z_n \sim \nu^{1-\alpha/2} (ar_n + b/r_n) / \sqrt{1 + 4\nu}$ where $\nu = \varepsilon n$ and

$$r_n = \left(\frac{1 + 2\nu + \sqrt{1 + 4\nu}}{\nu} \right)^{n-\alpha/2} \exp(\sqrt{1 + 4\nu}/(2\varepsilon))$$

4.65

b) $\gamma = k - n$ and $R \sim e^{\theta/\varepsilon}[R_0(n, k) + \dots]$, where $\theta = \kappa + (1 - \kappa) \ln(1 - \kappa)$ and $R_0 = A[(1 - \kappa)e^n]e^{n/2}$

Chapter 5

The Method of Homogenization

Introductory Example**5.3**

a) $\partial_x(\bar{D}\partial_x u_0) + g(u_0) = \partial_t u_0 + \langle f \rangle_\infty$

5.9

e) $\langle\langle D^{-1} \rangle\rangle_\infty^{-1} = 1/(\phi_\alpha D_\alpha + \phi_\beta D_\beta)$, which is a volume fraction weighted harmonic mean ; the harmonic mean is obtained only when $\phi_\alpha = \phi_\beta = 1/2$
 f) the greatest relative error occurs when $\phi_\alpha = 1/2$ with a relative absolute error of $(D_\alpha - D_\beta)^2/(4D_\alpha D_\beta)$; the error is zero if $\phi_\alpha = 0$ or if $\phi_\alpha = 1$

Porous Flow**5.19**

b) Setting $\mathbf{w}_q = (u_q, v_q, w_q)$ for $q = s, f \Rightarrow \nabla^2 \mathbf{w}_q = \mathbf{0}$ where \mathbf{w}_q is periodic and on the interface $D_s \mathbf{n} \cdot (\mathbf{e}_1 - \nabla_y u_s) = \alpha D_f \mathbf{n} \cdot (\mathbf{e}_1 - \nabla_y u_f)$, etc

5.20

b) $\nabla_y^2 - z^2 \alpha e^{z\phi_0} f = z \alpha e^{z\phi_0}$ where $\mathbf{n} \cdot \nabla_y f = 0$ on $\partial\Omega_f$
 e) ϕ satisfies the Poisson-Boltzmann equation $\nabla_y^2 \phi = z \alpha e^{a\phi}$ in $\partial\Omega_f$ where ϕ is periodic and $\mathbf{n} \cdot \nabla_y \phi = \sigma$ on $\partial\Omega_f$

Chapter 6

Introduction to Bifurcation and Stability

Introductory Example**6.1**

b) $y_\pm = \pm \frac{1}{3} \sqrt{1 - 4\lambda^2}$ for $-1 \leq 2\lambda \leq 1$ where y_+ is stable and y_- is unstable

6.3

b) supercritical pitchfork when $\lambda_n = (n\pi)^2$ and $\theta_n \sim 2\sqrt{2\varepsilon} \cos(n\pi x)/(n\pi)$
 b) $V_1 \sim -2\varepsilon^2/\pi^2 \Rightarrow \theta_s = 0$ is unstable for $\lambda > \pi^2$
 c) for θ_n one finds $V_n \sim -2\varepsilon^2/(n\pi)^2 \Rightarrow V_1 < V_2 < \dots < V_n < 0$ and this suggests that V_1 is the preferred configuration

6.4

- a) $y = A \sin(n\pi x)$ where $A = 0$ or $A^2 = 2(\lambda - (n\pi)^2)$; pitchfork bifurcation at $(\lambda_p, A_p) = ((n\pi)^2, 0)$ for $n = 1, 2, 3, \dots$
 b) $V_n = -A_n^4/2$

6.6

- a) $\theta = 0$ stable for $0 < \omega < 1$ and $\theta = \pm \arccos(\omega^{-2})$ is stable for $1 < \omega < \infty$; $\theta = \pm \pi$ is unstable

6.7

- c) $\kappa_0 = 0$ and $0 < \kappa_1 < 1/2$
 d) slope for $\Omega = 40$ is 24.34 and slope for $\Omega = 30$ is 20.36; $\omega = \varepsilon \Omega \sqrt{1/2 - \kappa_2}$

6.8

- c) $\omega = 2$ and $v \sim A(\tau) \cos(t + \theta(\tau))$ where $A = A_0 \sqrt{\alpha_0^2 + e^{-3\tau}} e^{3\tau/4}$ and $\theta = -\pi/4 + \arctan(\alpha_0 e^{3\tau/2})$

Relaxation Dynamics**6.10**

- b) $\ln(y_0) - \frac{1}{2}y_0^2 = t + \frac{1}{2}(\ln(3) - 3)$ and the corner is at $t = 1 - \frac{1}{2}\ln(3)$, $y = 1$, and $\nu = -2/3$
 d) $T \sim 3 - 2\ln(2) \approx 1.614$

6.13

- a) subcritical saddle-node at $(\lambda_b, y_b) \approx (0.1291, 0.0635)$

6.14

- a) subcritical saddle-node at $(\lambda_0, y_0) \approx (280.76, 2.360)$ and a subcritical saddle-node at $(\lambda_1, y_1) \approx (336.6, 6.6355)$
 c) $\tilde{Y} \sim y_1 - \sigma \varepsilon^{1/3} (a_0 Ai'(-\xi \tilde{\tau}) + b_0 Bi'(-\xi \tilde{\tau})) / (a_0 Ai(-\xi \tilde{\tau}) + b_0 Bi(-\xi \tilde{\tau}))$ where $\sigma = \xi / (15y_1 - 74)$, $\xi = [(15y_1 - 74)(y_1 - 1)]^{1/3}$, and $\tilde{\tau} = (\tau - \lambda_1) / \varepsilon^{2/3} \Rightarrow \tau_0 = \lambda_1 + (2.3381 \dots) \varepsilon^{2/3} / \xi$

6.15

- a) transcritical bifurcation at $(0, 0)$ and a subcritical saddle-node at $(9/8, 3/4)$
 c) $y \sim y_s + \varepsilon^{1/3} A \cos(\omega t + \theta)$ where $A_\infty^2 \{ \kappa^2 \omega^2 + \frac{9}{4} A_\infty^4 [-1 + 7(1 - 2y_s)^2 / \omega^2]^2 \} = 1$

An Example Involving A Nonlinear Partial Differential Equation**6.43**

- a) $u_s = \pm \sqrt{\varepsilon} A \sin(nx)$ and $\lambda_b = n^2$ where $A = 2/\sqrt{3}$
 b) $u_s = 0$ is stable for $\lambda < 1$ and $u_s = \pm \sqrt{\varepsilon} A \sin(x)$ is stable for $0 \leq \lambda - 1 \ll 1$

6.44

- a) $u_s = \varepsilon^\alpha A \sin(n\pi x)$ and $\lambda_b = (n\pi)^2$ where i) n odd $\Rightarrow A = 3/(8n\pi)$ and

$\alpha = 1$, and ii) n even $\Rightarrow A^2 = 6/(5\lambda_b)$ and $\alpha = 1/2$

b) stable for $\lambda < \pi^2$

Additional References: Bramson (1983) and Fisher (1937)

6.45

a) i) $u_s = 0, \forall \lambda$, and ii) $u_s = A_n \sin(nx)$ with $\lambda = \lambda_n(1 + A_n^2/8)$ where $\lambda_n = (n\pi)^2$

b) unbuckled state is stable if $\lambda < \pi^2$ and unstable if $\lambda > \pi^2$

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b) $\kappa = 0$ and stable for $\lambda < 1$

c) $v \sim \varepsilon[B_0 + A_0 \cos((1 + \frac{1}{2}\varepsilon)x + \theta_0)]$ and $\kappa \sim \frac{1}{4}A_0^2\varepsilon^2$ where $\varepsilon = \lambda - 1$

6.50

c) $u_0 = (u_i + u_j)/2, \alpha_{ij} = -(u_i - u_j)/2, \beta_{ij} = (u_j - u_i)/\sqrt{8}$ and $\chi_{ij} = \sqrt{2} \operatorname{sgn}(u_j - u_i)(u_1 + u_2 + u_3 - 3u_0)$; see Albano, et al. (1984)

6.51

b) for $0 \leq \lambda < 7$ the steady states are $u_i = \alpha_i x$, for $\alpha_1 < \alpha_2 < \alpha_3$ where u_1 and u_3 are stable, and u_2 is unstable; for $7 < \lambda$ the steady state u_3 is unique and stable

6.52

a) $u \sim u_s + \sum v_n e^{int} \Rightarrow \lambda < 1$

Bifurcation of Periodic Solutions

6.17

a) saddle-node at $(-1/4, -1/2)$, a Hopf at $(0, 0)$, and a transcritical at $(0, -1)$

b) $y \sim \sqrt{\varepsilon} A \cos(t_1 + \theta)$ where $8A' = A(4 - A^2)$ and $24\theta' = 36 - 31A^2$

6.18

a) $y_s = 0$ is asy stable onlt when $\lambda < 0$

b) $y \sim A(\varepsilon t) \sin(t + \theta_0)$ where $A(\tau) = 1/(\alpha_0 \alpha + c \exp(-n\tau))^{1/2n}$

6.20

c) Assuming $T = \frac{\pi}{2} + \varepsilon^2, t_2 = \varepsilon^2 t$, and $y \sim 1 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3 \dots$, shows that there is stable limit cycle for $\varepsilon > 0$. Note that $y_1 = a_1(t_2) \exp(i\lambda t_1) + cc$ where $\lambda > 0$.

6.34

b) $y = y_0$ stable for $\lambda y_0 < 2$ and there's a Hopf bifurcation at $\lambda y_0 = 2$

Systems of Ordinary Differential Equations

6.25

a) $r' = -r(r^2 - \lambda(1 - \lambda))$ and $\theta' = -1$

6.26

a) $y = v = (1 - \alpha - \beta + \mu)/2$

6.28

- a) $y = v = 1$ is stable for $\lambda < 1/(\gamma - 1)$ and unstable for $\lambda > 1/(\gamma - 1)$
 b) a Hopf bifurcation takes place

6.29

- a) $x = \gamma/(1 - \gamma)$, $n^2 = (1 - \gamma)/\alpha$ is stable for $(1 - \gamma)^3 < 4\alpha$

6.30

- a) $c_s = T_s \exp(-T_s)$ and $T_s = \mu/\kappa$; steady state is asymptotically stable if $\kappa > (T_s - 1) \exp(-T_s)$
 b) supercritical Hopf bifurcation at μ_1 and subcritical Hopf bifurcation at μ_r
 d) $\mu_1 \sim \kappa(1 + e\kappa + e^2\kappa^2 + \dots)$ and $\mu_r \sim \kappa(\zeta - \zeta^{-1} - 1.5\zeta^{-2} + \dots)$ where $\zeta > 1$ satisfies $\zeta \exp(-\zeta) = \kappa \Rightarrow \zeta \sim z_0 + \ln(z_0) + \ln(z_0)/z_0$ where $z_0 = -\ln(\kappa)$

6.31

- a) $x_s = \mu/\alpha$, $y_s = \alpha/\mu$ is stable
 b) $x_s = \mu/\alpha$, $y_s = \alpha/\mu$ is stable if $\mu > \mu_c$ and it's unstable if $0 < \mu < \mu_c$ where $\mu_c = \alpha\sqrt{1 - \alpha}$

6.35

- a) $\theta = z = 0$
 b) $\theta = 0$ and $z = \varepsilon\alpha_0 \cos(\kappa t)$; the motion is straight up and down; see van der Burgh(1968)

6.36

- a) $\lambda < 0$
 b) $k^2 = \lambda - r_0^2$ and $0 < r_0 < \sqrt{\lambda}$

6.37

- a) Hopf bifurcation along line $\lambda = -\mu$ for $-1 < \mu < 1$
 b) $y \sim \sqrt{\varepsilon}y_0(t, \tau)$ and $v \sim \sqrt{\varepsilon}v_0(t, \tau)$ where $\tau = \varepsilon t$. As $t \rightarrow \infty$, $y_0 \rightarrow \sin(\Theta)$ and $v_0 \rightarrow \sqrt{1 - \mu^2} \cos(\Theta) + \mu \sin(\Theta)$ where $\Theta = (1 - \varepsilon\mu)\sqrt{1 - \mu^2}t + \theta_0$

Weakly Coupled Nonlinear Oscillators**6.40**

- a) $\phi = 2 \arctan\left(\gamma + \text{sign}(cb)\sqrt{1 + \gamma^2} \tanh(q)\right)$, where $q = \frac{1}{2}c_b t \sqrt{1 + \gamma^2} + c_0$,
 $\gamma = c_a/c_b$, $c_b = b \left(\frac{\kappa_1}{\kappa_2} - \frac{\kappa_2}{\kappa_1}\right)$, $c_a = a \left(\frac{\kappa_2}{\kappa_1} + \frac{\kappa_1}{\kappa_2}\right)$

Metastability**6.55**

- c) $x_s = \frac{1}{4}(1 + 2x_0)$.