Summary: Minimization (Chapter 9)

Problem and Notation: minimize $f(\mathbf{x})$, and let $\mathbf{g} = \nabla f$

Basic Iterative Method

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{z}_k, \quad \text{for} \;\; k = 1, 2, 3, \ldots.$$

Newton: Take $\alpha_k = 1$ and $\mathbf{H}_k \mathbf{z}_k = -\mathbf{g}_k$, where $\mathbf{H}$ is the Hessian matrix of $f(\mathbf{x})$

Gradient Descent: $\mathbf{z}_1 = -\mathbf{g}_1$, otherwise

SDM: $\mathbf{z}_k = -\mathbf{g}_k$

CGM: $\mathbf{z}_k = -\mathbf{g}_k + \beta_{k-1}\mathbf{z}_{k-1}$, where $\beta_k = \dfrac{\mathbf{g}_{k+1} \bullet (\mathbf{g}_{k+1} - \mathbf{g}_k)}{\mathbf{g}_k \bullet \mathbf{g}_k}$

α_k: a line search is used to obtain $f(\mathbf{x}_k + \alpha_k \mathbf{z}_k) < f(\mathbf{x}_k)$

Special Case: if $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{x} \bullet \mathbf{b}$, where $\mathbf{A}$ is symmetric and positive definite, then

$$\alpha_k = \frac{\mathbf{g}_k \bullet \mathbf{g}_k}{\mathbf{g}_k \bullet \mathbf{A}\mathbf{g}_k} \qquad \text{and} \qquad \beta_k = \frac{\mathbf{g}_{k+1} \bullet \mathbf{g}_{k+1}}{\mathbf{g}_k \bullet \mathbf{g}_k}$$

Levenberg-Marquardt: Assume $f(\mathbf{x}) = \mathbf{f} \bullet \mathbf{f}$. Take $\alpha_k = 1$ and $\mathbf{S}_k \mathbf{z}_k = -\mathbf{g}_k$, where

$$\begin{aligned} \mathbf{S}_k &= \mathbf{G}_k + \mu_k \mathbf{D}_k, \\ \mathbf{D}_k &= \mathrm{diag}(\mathbf{G}_k), \\ \mathbf{G}_k &= 2\mathbf{J}_k^T \mathbf{J}_k \end{aligned}$$

Also, $\mathbf{J}$ is the Jacobian matrix for $\mathbf{f}(\mathbf{x})$ and μ_k is specified (e.g., $\mu_1 = 10$, $\mu_k = \frac{1}{2}\mu_{k-1}$).