Conjugate Gradient Method

(1) picking $\mathbf{x}_1$, set $\mathbf{r}_1 = \mathbf{b} - \mathbf{A}\mathbf{x}_1$ and $\mathbf{d}_1 = \mathbf{r}_1$

(2) for $k = 1, 2, 3, \cdots$

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{x}_k + \alpha_k \mathbf{d}_k \\ \mathbf{r}_{k+1} &= \mathbf{r}_k - \alpha_k \mathbf{q}_k \\ \mathbf{d}_{k+1} &= \mathbf{r}_{k+1} + \beta_k \mathbf{d}_k \end{aligned}$$

where

$$\begin{aligned} \mathbf{q}_k &= \mathbf{A}\mathbf{d}_k \\ \alpha_k &= \frac{\mathbf{r}_k \cdot \mathbf{r}_k}{\mathbf{d}_k \cdot \mathbf{q}_k} \\ \beta_k &= \frac{\mathbf{r}_{k+1} \cdot \mathbf{r}_{k+1}}{\mathbf{r}_k \cdot \mathbf{r}_k} \end{aligned}$$

Table 1: Outline of the conjugate gradient method (CGM) used to solve the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ is an $n \times n$ symmetric positive definite matrix.

Preconditioned Conjugate Gradient Method

(0) pick a symmetric positive definite matrix $\mathbf{M}$

(1) picking $\mathbf{x}_1$, set $\mathbf{r}_1 = \mathbf{b} - \mathbf{A}\mathbf{x}_1$, solve

$\mathbf{M}\mathbf{z}_1 = \mathbf{r}_1$ and then let $\mathbf{d}_1 = \mathbf{z}_1$

(2) for $k = 1, 2, 3, \cdots$

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{x}_k + \alpha_k \mathbf{d}_k \\
\mathbf{r}_{k+1} &= \mathbf{r}_k - \alpha_k \mathbf{q}_k \\
\mathbf{M}\mathbf{z}_{k+1} &= \mathbf{r}_{k+1} \\
\mathbf{d}_{k+1} &= \mathbf{z}_{k+1} + \beta_k \mathbf{d}_k
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{q}_k &= \mathbf{A}\mathbf{d}_k \\
\alpha_k &= \frac{\mathbf{z}_k \cdot \mathbf{r}_k}{\mathbf{d}_k \cdot \mathbf{q}_k} \\
\beta_k &= \frac{\mathbf{r}_{k+1} \cdot \mathbf{z}_{k+1}}{\mathbf{r}_k \cdot \mathbf{z}_k}
\end{aligned}
$$

Table 2: Outline of the preconditioned conjugate gradient method (PCGM) used to solve the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ is an $n \times n$ symmetric positive definite matrix.