Derivation of the Wave Equation

The objective is to provide an explanation of how the wave equation applies to the vibration of an elastic string. This will be done using the mass-spring material developed in Section 3.8 of the textbook, and a little trigonometry.

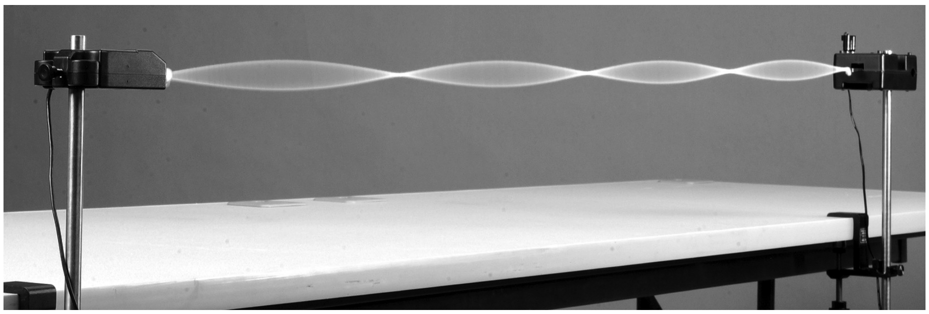
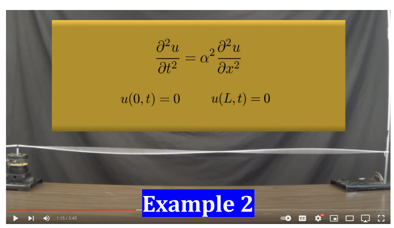
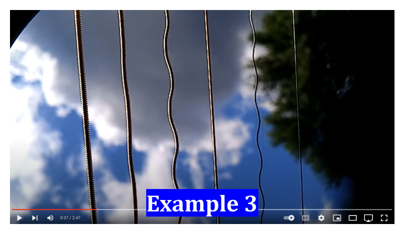


Figure 1: Time-lapsed photograph of a vibrating string. It is vibrating at approximately the fourth harmonic, or natural frequency, of the string. This solution is given in equation (7.68) of the textbook, with $n = 4$.

There are numerous online videos of vibrating strings. Here are three examples (the images are linked to the respective video):





(I) Modeling a String using Atoms and Springs

An unstretched elastic string can be thought of as made up of a row of atoms, each a distance ℓ_0 apart. Adjacent atoms in this case are connected by a spring as illustrated in Figure 2. These springs account for the forces between adjacent atoms. Also, for simplicity, a four atom string is shown. A more realistic depiction would have millions of atoms, and the atomic spacing ℓ_0 would be microscopic. This fact will be used later to simplify the formulas that are derived.

Assume that the string is stretched, with one end held at $x = 0$ and the other held at, say, $x = 1$ (see Figure 2). The atoms in the string are now a distance ℓ apart, where $\ell > \ell_0$. This is the position of the string before it is set into motion.

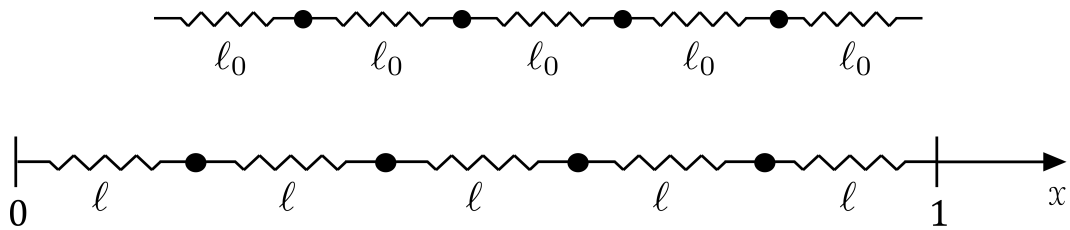


Figure 2: Upper: Unstretched row of atoms and springs making up the string. Lower: Spacing of atoms and springs after stretching the string so it occupies the interval $0 \le x \le 1$.

As usual, we assume that the restoring force in each spring is proportional to how far it is stretched. So, after stretching the string as described above, the resulting restoring force in each spring is

$$F = E\frac{\ell - \ell_0}{\ell_0} \,. \tag{1}$$

This formula differs slightly from Hooke's Law used in Section 3.8 in that we are now normalizing the stretch distance $\ell - \ell_0$ by the original length ℓ_0. A material for which this holds is said to be linearly elastic, and the positive constant E is called the Young's modulus for the material.

(II) Vertical Motion

It is assumed that as the string moves, the atoms in the string move vertically up and down. This is illustrated in Figure 3 in the case that the atoms have moved vertically upward. According to (1), the force F_b in the spring on the right, and the force F_a in the spring on the left, are

$$F_b = E\frac{\ell_b - \ell_0}{\ell_0} \,,$$

$$F_a = E\frac{\ell_a - \ell_0}{\ell_0} \,.$$

The respective force vectors are $\mathbf{F}_b = F_b(\cos\theta_b, \sin\theta_b)$ and $\mathbf{F}_a = -F_a(\cos\theta_a, \sin\theta_a)$.

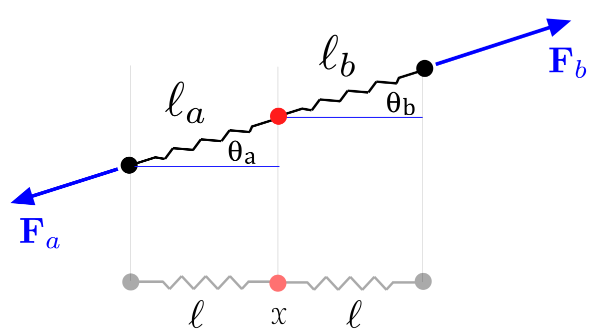


Figure 3: Stretched springs between atoms as the atoms move vertically. The rest position of each atom is also shown (using faded colors).

(III) Some Trigonometry

The vertical displacement of the red atom in Figure 3 is denoted as $u(x,t)$. The displacement of the atom on the left is $u(x-\ell,t)$, and the displacement of the one on the right is $u(x+\ell,t)$. From this we get:

$$\cos\theta_b = \frac{\ell}{\ell_b}\,, \quad \sin\theta_b = \frac{u(x+\ell,t)-u(x,t)}{\ell_b}\,,$$
$$\cos\theta_a = \frac{\ell}{\ell_a}\,, \quad \text{and} \quad \sin\theta_a = \frac{u(x,t)-u(x-\ell,t)}{\ell_a}\,.$$

Also,

$$\ell_b = \sqrt{\ell^2 + \big(u(x+\ell,t)-u(x,t)\big)^2}\,, \quad \text{and} \quad \ell_a = \sqrt{\ell^2 + \big(u(x,t)-u(x-\ell,t)\big)^2}\,.$$

(IV) Newton's Second Law: $\mathbf{F} = m\mathbf{a}$

In this problem $\mathbf{F} = \mathbf{F}_a + \mathbf{F}_b$ and, since the atoms only move in the vertical direction, $\mathbf{a} = (0, u_{tt})$. So, from the first component of $\mathbf{F} = m\mathbf{a}$ we conclude that $F_b\cos\theta_b - F_a\cos\theta_a = 0$. From this, and the formulas in (II) and (III), it follows that $F_a\ell_b = F_b\ell_a$. This means that the second component of $\mathbf{F}$ is

$$\begin{aligned} F_b\sin\theta_b - F_a\sin\theta_a &= F_b\,\frac{u(x+\ell,t)-u(x,t)}{\ell_b} - F_a\,\frac{u(x,t)-u(x-\ell,t)}{\ell_a} \\ &= F_b\,\frac{u(x+\ell,t)-2u(x,t)+u(x-\ell,t)}{\ell_b}\,. \end{aligned} \tag{2}$$

We now use the fact that the atomic spacing is much smaller that the overall length of the string, which means that $\ell \ll 1$. So, using Taylor's theorem,

$$\begin{aligned} u(x+\ell,t) &= u(x,t) + \ell u_x(x,t) + \frac{1}{2}\ell^2 u_{xx}(x,t) + \frac{1}{6}\ell^3 u_{xxx}(x,t) + \cdots \\ &\approx u(x,t) + \ell u_x(x,t) + \frac{1}{2}\ell^2 u_{xx}(x,t). \end{aligned}$$

Similarly, $u(x-\ell,t) \approx u(x,t) - \ell u_x(x,t) + \frac{1}{2}\ell^2 u_{xx}(x,t)$, and

$$\begin{aligned} \ell_b &= \sqrt{\ell^2 + (u(x+\ell,t)-u(x,t))^2} \\ &\approx \sqrt{\ell^2 + \ell^2 u_x(x,t)^2} = \ell\sqrt{1+u_x^2}\,. \end{aligned} \tag{3}$$

Substituting these into (2) yields

$$F_b\sin\theta_b - F_a\sin\theta_a \approx F_b\,\ell\frac{u_{xx}}{\sqrt{1+u_x^2}}\,.$$

Also, from (3), we get

$$F_b = E\frac{\ell_b - \ell_0}{\ell_0} \approx E\Big(\lambda\sqrt{1+u_x^2} - 1\Big)\,,$$

where $\lambda = \ell/\ell_0$ is a constant satisfying $\lambda > 1$.

The remaining term to find is the mass m. This is the mass of the string segment associated with the red atom in Figure 3. If the string is made of a material that has a mass density ρ, and if the original unstretched string segment is a cylinder with a cross-section area of A, then $m = \ell_0 A\rho$.

Combining the above results, $\mathbf{F} = m\mathbf{a}$ reduces to the nonlinear partial differential equation

$$\frac{E\lambda}{A\rho}\left(\lambda\sqrt{1+u_x^2}-1\right)\frac{u_{xx}}{\sqrt{1+u_x^2}} = u_{tt}\,. \tag{4}$$

(V) Final Assumption

The last assumption is that the string's amplitude is small when compared to distances along the x-axis over which its amplitude changes. More specifically, it's assumed that $u_x^2 \ll 1$ so that $\sqrt{1+u_x^2} \approx 1$. The PDE (4) then reduces to the wave equation

$$c^2 u_{xx} = u_{tt}\,, \tag{5}$$

where $c^2 = E\lambda(\lambda-1)/(\rho A)$ is a positive constant.

(VI) Summary

The principal assumptions made in the above derivation are:

(a) The string is linearly elastic.

(b) The string, before starting to move, is under tension.

(c) The points in the string move vertically.

(d) The string's motion has a relatively small amplitude.

(VII) Some Questions You Might Have

1. Does (5) really apply to the string in Figure 1?

Answer: The amplitude of that string is not particularly small, so assumption (d) above is questionable. However, in the experiment the string behaved in a manner consistent with (5). For example, the observed natural frequencies of the string have values accurately predicted using the formula given in (7.69) of the text. This is an indication that, even though the assumptions are not perfect, (5) provides a reasonable approximation to the motion of that string.

2. We used Hooke's law, or the slightly modified version in (1). How accurate is this assumption?

Answer: It depends on what the string (or spring) is made of. To illustrate, in Figure 4 the experimentally measured value of the force F in different types of guitar strings is given as a function of the stretch $\ell - \ell_0$. Hooke's law states that the force is a linear function of $\ell - \ell_0$. You see that each curve is approximately linear for a small stretch but not so for a large stretch.

To account for this, write (1) as $F = E\varepsilon$, where $\varepsilon = (\ell - \ell_0)/\ell_0$. So, if you want to be able to include larger stretches, then for an E-string you might assume that $F = E_1\varepsilon + E_2\varepsilon^3$ or maybe $F = E(e^\varepsilon - 1)$. Needless to say, the PDE becomes considerably more ~~difficult~~ interesting to solve when using a nonlinear function of ε.

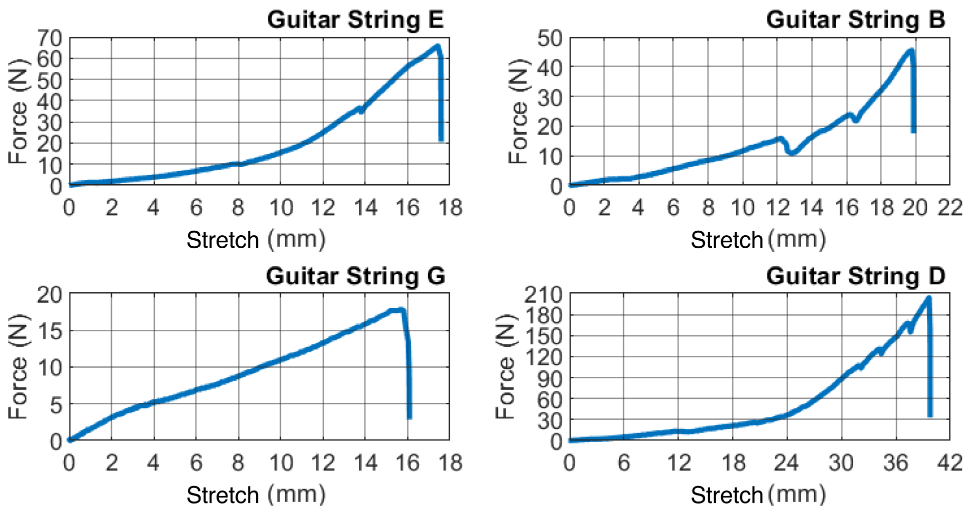


Figure 4: Restoring force measured in different types of guitar strings as you stretch them.